Harvard-MIT Mathematics Tournament 1999

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by djmathman, Binomial-theorem

–	Advanced Topics
1	One of the receipts for a math tournament showed that 72 identical trophies were purchased for $\$$-99.9-, where the first and last digits were illegible. How much did each trophy cost?
2	Stacy has d dollars. She enters a mall with 10 shops and a lottery stall. First she goes to the lottery and her money is doubled, then she goes into the first shop and spends 1024 dollars. After that she alternates playing the lottery and getting her money doubled (Stacy always wins) then going into a new shop and spending $\$1024$. When she comes out of the last shop she has no money left. What is the minimum possible value of d?
3	An unfair coin has the property that when flipped four times, it has the same nonzero probability of turning up 2 heads and 2 tails (in any order) as 3 heads and 1 tail (in any order). What is the probability of getting a head in any one flip?
4	You are given 16 pieces of paper numbered $16, 15, \ldots, 2, 1$ in that order. You want to put them in the order $1, 2, \ldots, 15, 16$ switching only two adjacent pieces of paper at a time. What is the minimum number of switches necessary?
5	For any finite set S, let $f(S)$ be the sum of the elements of S (if S is empty then $f(S) = 0$). Find the sum over all subsets E of S of $\dfrac{f(E)}{f(S)}$ for $S = \{1, 2, \cdots, 1999\}$.
6	Matt has somewhere between 1000 and 2000 pieces of paper he's trying to divide into piles of the same size (but not all in one pile or piles of one sheet each). He tries 2, 3, 4, 5, 6, 7, and 8 piles but ends up with one sheet left over each time. How many piles does he need?
7	Find an ordered pair (a, b) of real numbers for which $x^2 + ax + b$ has a non-real root whose cube is 343.
8	Let C be a circle with two diameters intersecting at an angle of 30 degrees. A circle S is tangent to both diameters and to C, and has radius 1. Find the largest possible radius of C.
9	As part of his e ffort to take over the world, Edward starts producing his own currency. As part of an eff ort to stop Edward, Alex works in the mint and produces 1 counterfeit coin for every 99 real ones. Alex isn't very good at this, so none of the counterfeit coins are the right weight.

Since the mint is not perfect, each coin is weighed before leaving. If the coin is not the right weight, then it is sent to a lab for testing. The scale is accurate 95% of the time, 5% of all the coins minted are sent to the lab, and the lab's test is accurate 90% of the time. If the lab says a coin is counterfeit, what is the probability that it really is?

10 Find the minimum possible value of the largest of xy, $1-x-y+xy$, and $x+y-2xy$ if $0 \le x \le y \le 1$.

– Algebra

1 If $a@b = \dfrac{a^3-b^3}{a-b}$, for how many real values of a does $a@1 = 0$?

2 For what single digit n does 91 divide the 9-digit number $12345n789$?

3 Alex is stuck on a platform floating over an abyss at 1 ft/s. An evil physicist has arranged for the platform to fall in (taking Alex with it) after traveling 100ft. One minute after the platform was launched, Edward arrives with a second platform capable of floating all the way across the abyss. He calculates for 5 seconds, then launches the second platform in such a way as to maximize the time that one end of Alex's platform is between the two ends of the new platform, thus giving Alex as much time as possible to switch. If both platforms are 5 ft long and move with constant velocity once launched, what is the speed of the second platform (in ft/s)?

4 Find all possible values of $\dfrac{d}{a}$ where $a^2-6ad+8d^2=0$, $a \neq 0$.

5 You are trapped in a room with only one exit, a long hallway with a series of doors and land mines. To get out you must open all the doors and disarm all the mines. In the room is a panel with 3 buttons, which conveniently contains an instruction manual. The red button arms a mine, the yellow button disarms two mines and closes a door, and the green button opens two doors. Initially 3 doors are closed and 3 mines are armed. The manual warns that attempting to disarm two mines or open two doors when only one is armed/closed will reset the system to its initial state. What is the minimum number of buttons you must push to get out?

6 Matt has somewhere between 1000 and 2000 pieces of paper he's trying to divide into piles of the same size (but not all in one pile or piles of one sheet each). He tries 2, 3, 4, 5, 6, 7, and 8 piles but ends up with one sheet left over each time. How many piles does he need?

7 Carl and Bob can demolish a building in 6 days, Anne and Bob can do it in 3, Anne and Carl in 5. How many days does it take all of them working together if Carl gets injured at the end of the first day and can't come back?

8 If $f(x)$ is a monic quartic polynomial such that $f(-1) = -1$, $f(2) = -4$, $f(-3) = -9$, and $f(4) = -16$, find $f(1)$.

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9 How many ways are there to cover a 3×8 rectangle with 12 identical dominoes?

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10 Pyramid $EARLY$ is placed in (x, y, z) coordinates so that $E = (10, 10, 0)$, $A = (10, -10, 0)$, $R = (-10, -10, 0)$, $L = (-10, 10, 0)$, and $Y = (0, 0, 10)$. Tunnels are drilled through the pyramid in such a way that one can move from (x, y, z) to any of the 9 points $(x, y, z-1)$, $(x \pm 1, y, z-1)$, $(x, y \pm 1, z-1)$, $(x \pm 1, y \pm 1, z-1)$. Sean starts at Y and moves randomly down to the base of the pyramid, choosing each of the possible paths with probability $\frac{1}{9}$. What is the probability that he ends up at the point $(8, 9, 0)$?

– Calculus

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1 Find all twice differentiable functions $f(x)$ such that $f''(x) = 0$, $f(0) = 19$, and $f(1) = 99$.

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2 A rectangle has sides of length $\sin x$ and $\cos x$ for some x. What is the largest possible area of such a rectangle?

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3 Find

$$\int_{-4\pi\sqrt{2}}^{4\pi\sqrt{2}} \left(\frac{\sin x}{1 + x^4} + 1 \right) dx.$$

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4 f is a continuous real-valued function such that $f(x+y) = f(x)f(y)$ for all real x, y. If $f(2) = 5$, find $f(5)$.

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5 Let $f(x) = x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{2x + \cdots}}}$. Find $f(99)f'(99)$.

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6 Evaluate $\frac{d}{dx}\left(\sin x - \frac{4}{3}\sin^3 x \right)$ when $x = 15$.

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7 If a right triangle is drawn in a semicircle of radius $1/2$ with one leg (not the hypotenuse) along the diameter, what is the triangle's maximum possible area?

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8 A circle is randomly chosen in a circle of radius 1 in the sense that a point is randomly chosen for its center, then a radius is chosen at random so that the new circle is contained in the original circle. What is the probability that the new circle contains the center of the original circle?

9 What fraction of the Earth's volume lies above the 45 degrees north parallel? You may assume the Earth is a perfect sphere. The volume in question is the smaller piece that we would get if the sphere were sliced into two pieces by a plane.

10 Let A_n be the area outside a regular n-gon of side length 1 but inside its circumscribed circle, let B_n be the area inside the n-gon but outside its inscribed circle. Find the limit as n tends to infinity of $\frac{A_n}{B_n}$.

– Geometry

– Oral

– Team