Harvard-MIT Mathematics Tournament 2000

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– Advanced Topics

1 How many different ways are there to paint the sides of a tetrahedron with exactly 4 colors? Each side gets its own color, and two colorings are the same if one can be rotated to get the other.

2 Simplify $\left(\frac{-1+i\sqrt{3}}{2}\right)^6+\left(\frac{-1-i\sqrt{3}}{2}\right)^6$ to the form $a+bi$.

3 Evaluate $\sum_{n=1}^{\infty}\frac{1}{n^2+2n}$.

4 Five positive integers from 1 to 15 are chosen without replacement. What is the probability that their sum is divisible by 3?

5 Find all 3-digit numbers which are the sums of the cubes of their digits.

6 6 people each have a hat. If they shuffle their hats and redistribute them, what is the probability that exactly one person gets their own hat back?

7 Assume that a, b, c, d are positive integers, and $\frac{a}{c}=\frac{b}{d}=\frac{3}{4}$, $\sqrt{a^2+c^2}-\sqrt{b^2+d^2}=15$. Find $ac+bd-ad-bc$.

8 How many non-isomorphic graphs with 9 vertices, with each vertex connected to exactly 6 other vertices, are there? (Two graphs are isomorphic if one can relabel the vertices of one graph to make all edges be exactly the same.)

9 The Cincinnati Reals are playing the Houston Alphas in the last game of the Swirled Series. The Alphas are leading by 1 run in the bottom of the 9th (last) inning, and the Reals are at bat. Each batter has a $\frac{1}{3}$ chance of hitting a single and a $\frac{2}{3}$ chance of making an out. If the Reals hit 5 or more singles before they make 3 outs, they will win. If the Reals hit exactly 4 singles before they make 3 outs, they will tie the game and send it into extra innings, and they will have a $\frac{3}{5}$ chance of eventually winning the game (since they have the added momentum of coming from behind). If the Reals hit fewer than 4 singles, they will LOSE! What is the probability that the Alphas hold off the Reals and win, sending the packed Alphadome into a frenzy? Express the answer as a fraction.

10 I call two people A and B and think of a natural number n. Then I give the number n to A and the number $n+1$ to B. I tell them that they have both been given natural numbers, and further that they are consecutive natural numbers. However, I don't tell A what B's number is and vice versa. I start by asing A if he knows B's number. He says "no", Then I ask B if he knows A's number, and he says "no" too. I go back to A and ask, and so on. A and B can both hear each other's responses. Do I ever get a "yes" in response? If so, who responds first with "yes" and how many times does he say "no" before this? Assume that both A and B are very intelligent and logical. You may need to consider multiple cases.

– Calculus

– General

– Geometry

– Guts

1 The sum of 3 real numbers is known to be zero. If the sum of their cubes is π^e, what is their product equal to?

2 If $X = 1 + x + x^2 + x^3 + \cdots$ and $Y = 1 + y + y^2 + y^3 + \cdots$, what is $1 + xy + x^2y^2 + x^3y^3 + \cdots$ in terms of X and Y only?

3 Using 3 colors, red, blue and yellow, how many different ways can you color a cube (modulo rigid rotations)?

4 Let ABC be a triangle and H be its orthocenter. If it is given that B is $(0, 0)$, C is $(1, 2)$ and H is $(5, 0)$, find A.

5 Find all natural numbers n such that n equals the cube of the sum of its digits.

6 If integers m, n, k satisfy $m^2 + n^2 + 1 = kmn$, what values can k have?

7 Suppose you are given a fair coin and a sheet of paper with the polynomial x^m written on it. Now for each toss of the coin, if heads show up, you must erase the polynomial x^r (where r is going to change with time - initially it is m) written on the paper and replace it with x^{r-1}. If tails show up, replace it with x^{r+1}. What is the expected value of the polynomial I get after m such tosses? (Note: this is a different concept from the most probable value)

8 Johny's father tells him: "I am twice as old as you will be seven years from the time I was thrice as old as you were". What is Johny's age?

9 A cubic polynomial f satisfies $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 4$. What is $f(5)$?

10 What is the total surface area of an ice cream cone, radius R, height H, with a spherical scoop of ice cream of radius r on top? (Given $R < r$)

11 Let M be the maximum possible value of $x_1x_2 + x_2x_3 + \cdots + x_5x_1$ where $x_1, x_2, \cdots x_5$ is a permutation of $(1, 2, 3, 4, 5)$ and let N be the number of permutations for which this maximum is attained. Evaluate $M + N$.

12 Calculate the number of ways of choosing 4 numbers from the set $1, 2, \cdots, 11$ such that at least 2 of the numbers are consecutive.

13 Determine the remainder when $(x^4 - 1)(x^2 - 1)$ is divided by $1 + x + x^2$.

14 $ABCD$ is a cyclic quadrilateral inscribed in a circle of radius 5, with $AB = 6$, $BC = 7$, $CD = 8$. Find AD.

15 Find the number of ways of filling a 8 by 8 grid with 0's and X's so that the number of 0's in each row and each column is odd.

16 Solve for real x, y: $x + y = 2$ $x^5 + y^5 = 82$

17 Find the highest power of 3 dividing $\binom{666}{333}$.

18 What is the value of $\sum_{n=1}^{\infty}(\tan^{-1}\sqrt{n} - \tan^{-1}\sqrt{n+1})$?

19 Define $a * b = \frac{a-b}{1-ab}$. What is $(1 * (2 * (3 * \cdots (n * (n + 1)) \cdots)))$?

20 What is the minimum possible perimeter of a triangle two of whose sides are along the x- and y-axes and such that the third contains the point $(1, 2)$?

21 How many ways can you color a necklace of 7 beads with 4 colors so that no two adjacent beads have the same color?

22 Find the smallest n such that 2^{2000} divides $n!$.

23 How many 7-digit numbers with distinct digits can be made that are divisible by 3?

24 At least how many moves must a knight make to get from one corner of a chessboard to the opposite corner?

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25 Find the next number in the sequence $131, 111311, 311321, 1321131211, \cdots$

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26 What are the last 3 digits of $1! + 2! + \cdots + 100!$

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27 What is the smallest number that can be written as a sum of 2 squares in 3 ways?

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28 What is the smallest possible volume to surface ratio of a solid cone with height = 1 unit?

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29 What is the value of $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}}$?

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30 $ABCD$ is a unit square. If $\angle PAC = \angle PCD$, find the length BP.

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31 Given collinear points A, B, C such that $AB = BC$. How can you construct a point D on AB such that $AD = 2DB$, using only a straightedge? (You are not allowed to measure distances)

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32 How many (nondegenerate) tetrahedrons can be formed from the vertices of an n-dimensional hypercube?

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33 Characterise all numbers that cannot be written as a sum of 1 or more consecutive odd numbers.

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34 What is the largest n such that $n! + 1$ is a square?

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35 If $1 + 2x + 3x^2 + ... = 9$, find x.

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36 If, in a triangle of sides a, b, c, the incircle has radius $\frac{b+c-a}{2}$, what is the magnitude of $\angle A$?

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37 A cone with semivertical angle 30° is half filled with water. What is the angle it must be tilted by so that water starts spilling?

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38 What is the largest number you can write with three 3s and three 8s, using only symbols $+, -, /, \times$ and exponentiation?

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39 If $x = \frac{1}{3}$, what is the value, rounded to 100 decimal digits, of $\sum_{n=0}^{7} \frac{2^n}{1+x^{2^n}}$?

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40 Let $\phi(n)$ denote the number of positive integers less than or equal to n and relatively prime to n. Find all natural numbers n and primes p such that $\phi(n) = \phi(np)$.

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41 A person observes a building of height h at an angle of inclination α from a point on the ground. After walking a distance a towards it, the angle is now 2α, and walking a further distance b causes it to increase to 3α. Find h in terms of a and b.

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42 A n by n magic square contains numbers from 1 to n^2 such that the sum of every row and every column is the same. What is this sum?

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43 Box A contains 3 black and 4 blue marbles. Box B has 7 black and 1 blue, whereas Box C has 2 black, 3 blue and 1 green marble. I close my eyes and pick two marbles from 2 different boxes. If it turns out that I get 1 black and 1 blue marble, what is the probability that the black marble is from box A and the blue one is from C?

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44 A function $f : \mathbb{Z} \implies \mathbb{Z}$ satisfies $f(x+4) - f(x) = 8x + 20$ $f(x^2 - 1) = (f(x) - x)^2 + x^2 - 2$ Find $f(0)$ and $f(1)$.

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45 Find all positive integers x for which there exists a positive integer y such that $\binom{x}{y} = 1999000$

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46 For what integer values of n is $1 + n + \frac{n^2}{2} + \cdots + \frac{n^n}{n!}$ an integer?

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47 Find an $n < 100$ such that $n \cdot 2^n - 1$ is prime. Score will be $n - 5$ for correct n, $5 - n$ for incorrect n (0 points for answer < 5)

– Power

– Team

– Algebra

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1 How many integers x satisfy $|x| + 5 < 7$ and $|x - 3| > 2$?

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2 Evaluate $2000^3 - 1999 \cdot 2000^2 - 1999^2 \cdot 2000 + 1999^3$

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3 Five students take a test on which any integer score from 0 to 100 inclusive is possible. What is the largest possible difference between the median and the mean of the scores?

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4 What is the fewest number of multiplications required to reach x^{2000} from x, using only previously generated powers of x? For example $x \to x^2 \to x^4 \to x^8 \to x^{16} \to x^{32} \to x^{64} \to x^{128} \to x^{256} \to x^{512} \to x^{1024} \to x^{1536} \to x^{1792} \to x^{1920} \to x^{1984} \to x^{2000}$ uses 15 multiplications.

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5 A jacket was originally priced $\$100$. The price was reduced by 10% three times and increased by 10% four times in some order. To the nearest cent, what was the final price?

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6 Barbara, Edward, Abhinav, and Alex took turns writing this test. Working alone, they could finish it in $10, 9, 11$, and 12 days, respectively. If only one person works on the test per day, and nobody works on it unless everyone else has spent at least as many days working on it, how many days (an integer) did it take to write this test?

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7 A number n is called multiplicatively perfect if the product of all the positive divisors of n is n^2. Determine the number of positive multiplicatively perfect numbers less than 100.

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8 A man has three daughters. The product of their ages is 168, and he remembers that the sum of their ages is the number of trees in his yard. He counts the trees but cannot determine any of their ages. What are all possible ages of his oldest daughter?

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9 $\frac{a}{c} = \frac{b}{d} = \frac{3}{4}$, $\sqrt{a^2+c^2} - \sqrt{b^2+d^2} = 15$. Find $ac + bd - ad - bc$.

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10 Find the smallest positive integer a such that $x^4 + a^2$ is not prime for any integer x.

– Oral

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1 Find all integer solutions to $m^2 = n^6 + 1$.

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2 How many positive solutions are there to $x^{10} + 7x^9 + 14x^8 + 1729x^7 - 1379x^6 = 0$? How many positive integer solutions?

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3 Suppose the positive integers a, b, c satisfy $a^n + b^n = c^n$, where n is a positive integer greater than 1. Prove that $a, b, c > n$.
(Note: Fermat's Last Theorem may *not* be used)

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4 On an n by n chessboard, numbers are written on each square so that the number in a square is the average of the numbers on the adjacent squares. Show that all the numbers are the same.

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5 Show that it is impossible to find a triangle in the plane with all integer coordinates such that the lengths of the sides are all odd.

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6 Prove that every multiple of 3 can be written as a sum of four cubes (positive or negatives).

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7 A regular tetrahedron of volume 1 is filled with water of total volume $\frac{7}{16}$. Is it possible that the center of the tetrahedron lies on the surface of the water? How about in a cube of volume 1?

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8 Let $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ and $\vec{v_5}$ be vectors in three dimensions. Show that for some i, j in $1, 2, 3, 4, 5$, $\vec{v_i} \cdot \vec{v_j} \geq 0$.

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9 f is a polynomial of degree n with integer coefficients and $f(x) = x^2 + 1$ for $x = 1, 2, \cdot, n$. What are the possible values for $f(0)$?

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10 23 frat brothers are sitting in a circle. One, call him Alex, starts with a gallon of water. On the first turn, Alex gives each person in the circle some rational fraction of his water. On each subsequent turn, every person with water uses the same scheme as Alex did to distribute his water, but in relation to themselves. For instance, suppose Alex gave $\frac{1}{2}$ and $\frac{1}{6}$ of his water to his left and right neighbors respectively on the first turn and kept $\frac{1}{3}$ for himself. On each subsequent turn everyone gives $\frac{1}{2}$ and $\frac{1}{6}$ of the water they started the turn with to their left and right neighbors, respectively, and keep the final third for themselves. After 23 turns, Alex again has a gallon of water. What possibilities are there for the scheme he used in the first turn? (Note: you may find it useful to know that $1 + x + x^2 + \cdot + x^{23}$ has no polynomial factors with rational coefficients)
