Harvard-MIT Mathematics Tournament 2004

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–	Algebra
1	How many ordered pairs of integers (a, b) satisfy all of the following inequalities? $$\begin{aligned} a^2+b^2 &< 16 \\ a^2+b^2 &< 8a \\ a^2+b^2 &< 8b \end{aligned}$$
2	Find the largest number n such that $(2004!)!$ is divisible by $((n!)!)!$.
3	Compute $$\left\lfloor \frac{2005^3}{2003\cdot 2004} - \frac{2003^3}{2004\cdot 2005} \right\rfloor$$
4	Evaluate the sum $$\frac{1}{2\lfloor\sqrt{1}\rfloor+1} + \frac{1}{2\lfloor\sqrt{2}\rfloor+1} + \frac{1}{2\lfloor\sqrt{3}\rfloor+1} + \cdots + \frac{1}{2\lfloor\sqrt{100}\rfloor+1}$$
5	There exists a positive real number x such that $\cos(\arctan(x)) = x$. Find the value of x^2.
6	Find all real solutions to $x^4 + (2-x)^4 = 34$.
7	If x, y, k are positive reals such that $$3 = k^2\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right) + k\left(\frac{x}{y}+\frac{y}{x}\right),$$ find the maximum possible value of k.
8	Let x be a real number such that $x^3 + 4x = 8$. Determine the value of $x^7 + 64x^2$.

9 A sequence of positive integers is defined by $a_0 = 1$ and $a_{n+1} = a_n^2 + 1$ for each $n \geq 0$. Find gcd(a_{999}, a_{2004}).

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10 There exists a polynomial P of degree 5 with the following property: if z is a complex number such that $z^5 + 2004z = 1$, then $P(z^2) = 0$. Calculate the quotient $\frac{P(1)}{P(-1)}$.

– Calculus

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1 Let $f(x) = \sin(\sin(x))$. Evaluate

$$\lim_{h \to 0} \frac{f(x+h) - f(h)}{x}$$

at $x = \pi$.

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2 Suppose the function $f(x) - f(2x)$ has derivative 5 at $x = 1$ and derivative 7 at $x = 2$. Find the derivative of $f(x) - f(4x)$ at $x = 1$.

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3 Find

$$\lim_{x \to \infty} \left(\sqrt[3]{x^3 + x^2} - \sqrt[3]{x^3 - x^2} \right).$$

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4 Let $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(x))))))))$, and suppose that the number a satisfies the equation $a = \cos a$. Express $f'(a)$ as a polynomial in a.

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5 A mouse is sitting in a toy car on a negligibly small turntable. The car cannot turn on its own, but the mouse can control when the car is launched and when the car stops (the car has brakes). When the mouse chooses to launch, the car will immediately leave the turntable on a straight trajectory at 1 meter per second. Suddenly someone turns on the turntable; it spins at 30 rpm. Consider the set S of points the mouse can reach in his car within 1 second after the turntable is set in motion. What is the area of S, in square meters?

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6 For $x > 0$, let $f(x) = x^x$. Find all values of x for which $f(x) = f'(x)$.

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7 Find the area of the region in the xy-plane satisfying $x^6 - x^2 + y^2 \leq 0$.

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8 If x and y are real numbers with $(x+y)^4 = x - y$, what is the maximum possible value of y?

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9 Find the positive constant c_0 such that the series

$$\sum_{n=0}^{\infty} \frac{n!}{(cn)^n}$$

converges for $c > c_0$ and diverges for $0 < c < c_0$.

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10 Let $P(x) = x^3 - \frac{3}{2}x^2 + x + \frac{1}{4}$. Let $P^{[1]}(x) = P(x)$, and for $n \geq 1$, let $P^{n+1}(x) = P^{[n]}(P(x))$. Evaluate:

$$\int_0^1 P^{[2004]}(x)\,\mathrm{d}x.$$

– Combinatorics

1 There are 1000 rooms in a row along a long corridor. Initially the first room contains 1000 people and the remaining rooms are empty. Each minute, the following happens: for each room containing more than one person, someone in that room decides it is too crowded and moves to the next room. All these movements are simultaneous (so nobody moves more than once within a minute). After one hour, how many different
rooms will have people in them?

2 How many ways can you mark 8 squares of an 8×8 chessboard so that no two marked squares are in the same row or column, and none of the four corner squares is marked? (Rotations and reflections are considered different.)

3 A class of 10 students took a math test. Each problem was solved by exactly 7 of the students. If the first nine students each solved 4 problems, how many problems did the tenth student solve?

4 Andrea flips a fair coin repeatedly, continuing until she either flips two heads in a row (the sequence $\mathtt{HH}$) or flips tails followed by heads (the sequence $\mathtt{TH}$). What is the probability that she will stop after flipping $\mathtt{HH}$?

5 A best-of-9 series is to be played between two teams; that is, the first team to win 5 games is the winner. The Mathletes have a chance of $\frac{2}{3}$ of winning any given game. What is the probability that exactly 7 games will need to be played to determine a winner?

6 A committee of 5 is to be chosen from a group of 9 people. How many ways can it be chosen, if Bill and Karl must serve together or not at all, and Alice and Jane refuse to serve with each other?

7 We have a polyhedron such that an ant can walk from one vertex to another, traveling only along edges, and traversing every edge exactly once. What is the smallest possible total number of vertices, edges, and faces of this polyhedron?

8 Urn A contains 4 white balls and 2 red balls. Urn B contains 3 red balls and 3 black balls. An urn is randomly selected, and then a ball inside of that urn is removed. We then repeat the process of selecting an urn and drawing out a ball, without returning the first ball. What is the probability that the first ball drawn was red, given that the second ball drawn was black?

9 A classroom consists of a 5×5 array of desks, to be filled by anywhere from 0 to 25 students, inclusive. No student will sit at a desk unless either all other desks in its row or all others in its column are filled (or both). Considering only the set of desks that are occupied (and not which student sits at each desk), how many possible arrangements are there?

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10 In a game similar to three card monte, the dealer places three cards on the table: the queen of spades and two red cards. The cards are placed in a row, and the queen starts in the center; the card configuration is thus RQR. The dealer proceeds to move. With each move, the dealer randomly switches the center card with one of the two edge cards (so the configuration after the first move is either RRQ or QRR). What is the probability that, after 2004 moves, the center card is the queen?

– General part 1

– General part 2

– Geometry

– Guts

– Team A

– Team B