Harvard-MIT Mathematics Tournament 2005

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- Algebra

1   How many real numbers x are solutions to the following equation?
$$|x - 1| = |x - 2| + |x - 3|$$

2   How many real numbers x are solutions to the following equation?
$$2003^x + 2004^x = 2005^x$$

3   Let x, y, and z be distinct real numbers that sum to 0. Find the maximum possible value of
$$\frac{xy + yz + zx}{x^2 + y^2 + z^2}.$$

4   If $a, b, c > 0$, what is the smallest possible value of $\left\lfloor \dfrac{a+b}{c} \right\rfloor + \left\lfloor \dfrac{b+c}{a} \right\rfloor + \left\lfloor \dfrac{c+a}{b} \right\rfloor$? (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

5   Ten positive integers are arranged around a circle. Each number is one more than the greatest common divisor of its two neighbors. What is the sum of the ten numbers?

6   Find the sum of the x-coordinates of the distinct points of intersection of the plane curves given by $x^2 = x + y + 4$ and $y^2 = y - 15x + 36$.

7   Let x be a positive real number. Find the maximum possible value of
$$\frac{x^2 + 2 - \sqrt{x^4 + 4}}{x}.$$

8   Compute
$$\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}.$$

9   The number $27,000,001$ has exactly four prime factors. Find their sum.

10   Find the sum of the absolute values of the roots of $x^4 - 4x^3 - 4x^2 + 16x - 8 = 0$.

–   Calculus

1   Let $f(x) = x^3 + ax + b$, with $a \neq b$, and suppose the tangent lines to the graph of f at $x = a$ and $x = b$ are parallel. Find $f(1)$.

2   A plane curve is parameterized by $x(t) = \int_t^\infty \frac{\cos u}{u} \, \mathrm{d}u$ and $y(t) = \int_t^\infty \frac{\sin u}{u} \, \mathrm{d}u$ for $1 \leq t \leq 2$. What is the length of the curve?

3   Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function with $\int_0^1 f(x) f'(x) \, \mathrm{d}x = 0$ and $\int_0^1 f(x)^2 f'(x) \, \mathrm{d}x = 18$. What is $\int_0^1 f(x)^4 f'(x) \, \mathrm{d}x$?

4   Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $f'(x)^2 = f(x) f''(x)$ for all x. Suppose $f(0) = 1$ and $f^{(4)}(0) = 9$. Find all possible values of $f'(0)$.

5   Calculate
$$\lim_{x \to 0^+} \left(x^{x^x} - x^x \right).$$

6   The graph of $r = 2 + \cos 2\theta$ and its reflection over the line $y = x$ bound five regions in the plane. Find the area of the region containing the origin.

7   Two ants, one starting at $(-1, 1)$, the other at $(1, 1)$, walk to the right along the parabola $y = x^2$ such that their midpoint moves along the line $y = 1$ with constant speed 1. When the left ant first hits the line $y = \frac{1}{2}$, what is its speed?

8   If f is a continuous real function such that $f(x - 1) + f(x + 1) \geq x + f(x)$ for all x, what is the minimum possible value of $\int_1^{2005} f(x) \, \mathrm{d}x$?

9   Compute
$$\sum_{k=0}^\infty \frac{4}{(4k)!}.$$

10 Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $f'(x) = f(1-x)$ for all x and $f(0) = 1$. Find $f(1)$.

– Combinatorics

– General Part 1

– General Part 2

– Geometry

– Guts

– Team A

– Team B