Harvard-MIT Mathematics Tournament 2006

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by djmathman

–	Algebra
1	Larry can swim from Harvard to MIT (with the current of the Charles River) in 40 minutes, or back (against the current) in 45 minutes. How long does it take him to row from Harvard to MIT, if he rows the return trip in 15 minutes? (Assume that the speed of the current and Larrys swimming and rowing speeds relative to the current are all constant.) Express your answer in the format mm:ss.
2	Find all real solutions (x, y) of the system $x^2 + y = 12 = y^2 + x$.
3	The train schedule in Hummut is hopelessly unreliable. Train A will enter Intersection X from the west at a random time between $9:00$ am and $2:30$ pm; each moment in that interval is equally likely. Train B will enter the same intersection from the north at a random time between $9:30$ am and $12:30$ pm, independent of Train A; again, each moment in the interval is equally likely. If each train takes 45 minutes to clear the intersection, what is the probability of a collision today?
4	Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Find $$\sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}}.$$
5	Tim has a working analog 12-hour clock with two hands that run continuously (instead of, say, jumping on the minute). He also has a clock that runs really slowat half the correct rate, to be exact. At noon one day, both clocks happen to show the exact time. At any given instant, the hands on each clock form an angle between $0°$ and $180°$ inclusive. At how many times during that day are the angles on the two clocks equal?
6	Let a, b, c be the roots of $x^3 - 9x^2 + 11x - 1 = 0$, and let $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Find $s^4 - 18s^2 - 8s$.
7	Let $$f(x) = x^4 - 6x^3 + 26x^2 - 46x + 65.$$ Let the roots of $f(x)$ be $a_k + ib_k$ for $k = 1, 2, 3, 4$. Given that the a_k, b_k are all integers, find $\|b_1\| + \|b_2\| + \|b_3\| + \|b_4\|$.

8 Solve for all complex numbers z such that $z^4 + 4z^2 + 6 = z$.

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9 Compute the value of the infinite series

$$\sum_{n=2}^{\infty} \frac{n^4 + 3n^2 + 10n + 10}{2^n \cdot (n^4 + 4)}$$

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10 Determine the maximum value attained by

$$\frac{x^4 - x^2}{x^6 + 2x^3 - 1}$$

over real numbers $x > 1$.

– Calculus

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1 A nonzero polynomial $f(x)$ with real coefficients has the property that $f(x) = f'(x)f''(x)$. What is the leading coefficient of $f(x)$?

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2 Compute $\lim_{x \to 0} \frac{e^{x \cos x} - 1 - x}{\sin(x^2)}$.

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3 At time 0, an ant is at $(1, 0)$ and a spider is at $(-1, 0)$. The ant starts walking counterclockwise around the unit circle, and the spider starts creeping to the right along the x-axis. It so happens that the ant's horizontal speed is always half the spider's. What will the shortest distance ever between the ant and the spider be?

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4 Compute $\sum_{k=1}^{\infty} \frac{k^4}{k!}$.

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5 Compute $\int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$.

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6 A triangle with vertices at $(1003, 0)$, $(1004, 3)$, and $(1005, 1)$ in the xy-plane is revolved all the way around the y-axis. Find the volume of the solid thus obtained.

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7 Find all positive real numbers c such that the graph of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - cx$ has the property that the circle of curvature at any local extremum is centered at a point on the x-axis.

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8 Compute $\int_0^{\pi/3} x \tan^2(x) dx$.

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9 Compute the sum of all real numbers x such that

$$2x^6 - 3x^5 + 3x^4 + x^3 - 3x^2 + 3x - 1 = 0.$$

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10 Suppose f and g are differentiable functions such that

$$xg(f(x))f'(g(x))g'(x) = f(g(x))g'(f(x))f'(x)$$

for all real x. Moreover, f is nonnegative and g is positive. Furthermore,

$$\int_0^a f(g(x))dx = 1 - \frac{e^{-2a}}{2}$$

for all reals a. Given that $g(f(0)) = 1$, compute the value of $g(f(4))$.

– Combinatorics

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1 Vernonia High School has 85 seniors, each of whom plays on at least one of the schools three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

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2 Compute

$$\sum_{n_{60}=0}^{2} \sum_{n_{59}=0}^{n_{60}} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} \sum_{n_0=0}^{n_1} 1.$$

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3 A moth starts at vertex A of a certain cube and is trying to get to vertex B, which is opposite A, in five or fewer steps, where a step consists in traveling along an edge from one vertex to another. The moth will stop as soon as it reaches B. How many ways can the moth achieve its objective?

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4 A dot is marked at each vertex of a triangle ABC. Then, 2, 3, and 7 more dots are marked on the sides AB, BC, and CA, respectively. How many triangles have their vertices at these dots?

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5 Fifteen freshmen are sitting in a circle around a table, but the course assistant (who remains standing) has made only six copies of todays handout. No freshman should get more than one handout, and any freshman who does not get one should be able to read a neighbors. If the freshmen are distinguishable but the handouts are not, how many ways are there to distribute the six handouts subject to the above conditions?

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6 For how many ordered triplets (a, b, c) of positive integers less than 10 is the product $a \times b \times c$ divisible by 20?

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7 Let n be a positive integer, and let Pushover be a game played by two players, standing squarely facing each other, pushing each other, where the first person to lose balance loses. At the HMPT, 2^{n+1} competitors, numbered 1 through 2^{n+1} clockwise, stand in a circle. They are equals in Pushover: whenever two of them face off, each has a 50% probability of victory. The tournament unfolds in $n+1$ rounds. In each rounjd, the referee randomly chooses one of the surviving players, and the players pair off going clockwise, starting from the chosen one. Each pair faces off in Pushover, and the losers leave the circle. What is the probability that players 1 and 2^n face each other in the last round? Express your answer in terms of n.

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8 In how many ways can we enter numbers from the set $\{1, 2, 3, 4\}$ into a 4×4 array so that all of the following conditions hold?

(a) Each row contains all four numbers.

(b) Each column contains all four numbers.

(c) Each ”quadrant” contains all four numbers. (The quadrants are the four corner 2×2 squares.)

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9 Eight celebrities meet at a party. It so happens that each celebrity shakes hands with exactly two others. A fan makes a list of all unordered pairs of celebrities who shook hands with each other. If order does not matter, how many different lists are possible?

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10 Somewhere in the universe, n students are taking a 10-question math competition. Their collective performance is called *laughable* if, for some pair of questions, there exist 57 students such that either all of them answered both questions correctly or none of them answered both questions correctly. Compute the smallest n such that the performance is necessarily laughable.

– General Part 1

– General Part 2

– Geometry

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1 Octagon $ABCDEFGH$ is equiangular. Given that $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$, and $EF = FG = 2$, compute the perimeter of the octagon.

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2 Suppose ABC is a scalene right triangle, and P is the point on hypotenuse $\overline{AC}$ such that $\angle ABP = 45°$. Given that $AP = 1$ and $CP = 2$, compute the area of ABC.

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3 Let A, B, C, and D be points on a circle such that $AB = 11$ and $CD = 19$. Point P is on segment AB with $AP = 6$, and Q is on segment CD with $CQ = 7$. The line through P and Q intersects the circle at X and Y. If $PQ = 27$, find XY.

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4 Let ABC be a triangle such that $AB = 2$, $CA = 3$, and $BC = 4$. A semicircle with its diameter on BC is tangent to AB and AC. Compute the area of the semicircle.

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5 Triangle ABC has side lengths $AB = 2\sqrt{5}$, $BC = 1$, and $CA = 5$. Point D is on side AC such that $CD = 1$, and F is a point such that $BF = 2$ and $CF = 3$. Let E be the intersection of lines AB and DF. Find the area of $CDEB$.

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6 A circle of radius t is tangent to the hypotenuse, the incircle, and one leg of an isosceles right triangle with inradius $r = 1 + \sin\frac{\pi}{8}$. Find rt.

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7 Suppose $ABCD$ is an isosceles trapezoid in which $\overline{AB} \parallel \overline{CD}$. Two mutually externally tangent circles ω_1 and ω_2 are inscribed in $ABCD$ such that ω_1 is tangent to $\overline{AB}$,$\overline{BC}$, and $\overline{CD}$ while ω_2 is tangent to $\overline{AB}$, $\overline{DA}$, and $\overline{CD}$. Given that $AB = 1$, $CD = 6$, compute the radius of either circle.

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8 Triangle ABC has a right angle at B. Point D lies on side BC such that $3\angle BAD = \angle BAC$. Given $AC = 2$ and $CD = 1$, compute BD.

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9 Four spheres, each of radius r, lie inside a regular tetrahedron with side length 1 such that each sphere is tangent to three faces of the tetrahedron and to the other three spheres. Find r.

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10 Triangle ABC has side lengths $AB = 65$, $BC = 33$, and $AC = 56$. Find the radius of the circle tangent to sides AC and BC and to the circumcircle of triangle ABC.

– Guts

– Team A

– Team B
