Harvard-MIT Mathematics Tournament 2008

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by worthawholebean

–	Algebra
–	February 23rd
1	Positive real numbers x, y satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy.
2	Let $f(n)$ be the number of times you have to hit the $\sqrt{\ }$ key on a calculator to get a number less than 2 starting from n. For instance, $f(2) = 1$, $f(5) = 2$. For how many $1 < m < 2008$ is $f(m)$ odd?
3	Determine all real numbers a such that the inequality $\lvert x^2 + 2ax + 3a\rvert \leq 2$ has exactly one solution in x.
4	The function f satisfies $$f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1$$ for all real numbers x, y. Determine the value of $f(10)$.
5	Let $f(x) = x^3 + x + 1$. Suppose g is a cubic polynomial such that $g(0) = -1$, and the roots of g are the squares of the roots of f. Find $g(9)$.
6	A *root of unity* is a complex number that is a solution to $z^n = 1$ for some positive integer n. Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers a and b.
7	Compute $\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{k}{2^{n+k}}$.
8	Compute $\arctan(\tan 65^\circ - 2\tan 40^\circ)$. (Express your answer in degrees.)
9	Let S be the set of points (a, b) with $0 \leq a, b \leq 1$ such that the equation $$x^4 + ax^3 - bx^2 + ax + 1 = 0$$ has at least one real root. Determine the area of the graph of S.
10	Evaluate the infinite sum $$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}.$$

–	Calculus
–	February 23rd
1	Let $f(x) = 1 + x + x^2 + \cdots + x^{100}$. Find $f'(1)$.
2	**(3)** Let ℓ be the line through $(0,0)$ and tangent to the curve $y = x^3 + x + 16$. Find the slope of ℓ.
3	**(4)** Find all $y > 1$ satisfying $\int_1^y x \ln x \, dx = \frac{1}{4}$.
4	**(4)** Let a, b be constants such that $\lim_{x \to 1} \frac{(\ln(2-x))^2}{x^2+ax+b} = 1$. Determine the pair (a,b).
5	**(4)** Let $f(x) = \sin^6\left(\frac{x}{4}\right) + \cos^6\left(\frac{x}{4}\right)$ for all real numbers x. Determine $f^{(2008)}(0)$ (i.e., f differentiated 2008 times and then evaluated at $x = 0$).
6	Determine the value of $\lim_{n\to\infty} \sum_{k=0}^n \binom{n}{k}^{-1}$.
7	**(5)** Find p so that $\lim_{x\to\infty} x^p \left(\sqrt[3]{x+1} + \sqrt[3]{x-1} - 2\sqrt[3]{x}\right)$ is some non-zero real number.
8	Let $T = \int_0^{\ln 2} \frac{2e^{3x}+e^{2x}-1}{e^{3x}+e^{2x}-e^x+1} dx$. Evaluate e^T.
9	**(7)** Evaluate the limit $\lim_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)} \left(1^1 \cdot 2^2 \cdot \cdots \cdot n^n\right)^{\frac{1}{n^2}}$.
10	**(8)** Evaluate the integral $\int_0^1 \ln x \ln(1-x) \, dx$.
–	Combinatorics
–	February 23rd
1	A $3 \times 3 \times 3$ cube composed of 27 unit cubes rests on a horizontal plane. Determine the number of ways of selecting two distinct unit cubes from a $3 \times 3 \times 1$ block (the order is irrelevant) with the property that the line joining the centers of the two cubes makes a 45° angle with the horizontal plane.
2	Let $S = \{1, 2, \ldots, 2008\}$. For any nonempty subset $A \in S$, define $m(A)$ to be the median of A (when A has an even number of elements, $m(A)$ is the average of the middle two elements). Determine the average of $m(A)$, when A is taken over all nonempty subsets of S.

3 Farmer John has 5 cows, 4 pigs, and 7 horses. How many ways can he pair up the animals so that every pair consists of animals of different species? (Assume that all animals are distinguishable from each other.)

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4 Kermit the frog enjoys hopping around the innite square grid in his backyard. It takes him 1 Joule of energy to hop one step north or one step south, and 1 Joule of energy to hop one step east or one step west. He wakes up one morning on the grid with 100 Joules of energy, and hops till he falls asleep with 0 energy. How many different places could he have gone to sleep?

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5 Let S be the smallest subset of the integers with the property that $0 \in S$ and for any $x \in S$, we have $3x \in S$ and $3x+1 \in S$. Determine the number of non-negative integers in S less than 2008.

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6 A Sudoku matrix is dened as a 9×9 array with entries from $\{1, 2, \ldots, 9\}$ and with the constraint that each row, each column, and each of the nine 3×3 boxes that tile the array contains each digit from 1 to 9 exactly once. A Sudoku matrix is chosen at random (so that every Sudoku matrix has equal probability of being chosen). We know two of the squares in this matrix, as shown. What is the probability that the square marked by ? contains the digit 3?

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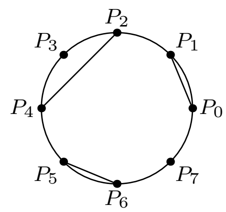
7 Let $P_1, P_2, \ldots, P_8$ be 8 distinct points on a circle. Determine the number of possible configurations made by drawing a set of line segments connecting pairs of these 8 points, such that: (1) each P_i is the endpoint of at most one segment and (2) no two segments intersect. (The configuration with no edges drawn is allowed. An example of a valid configuration is shown below.)



8 Determine the number of ways to select a sequence of 8 sets $A_1, A_2, \ldots, A_8$, such that each is a subset (possibly empty) of $\{1,2\}$ and A_m contains A_n if m divides n.

9 On an innite chessboard (whose squares are labeled by (x,y), where x and y range over all integers), a king is placed at $(0,0)$. On each turn, it has probability of 0.1 of moving to each of the four edge-neighboring squares, and a probability of 0.05 of moving to each of the four diagonally-neighboring squares, and a probability of 0.4 of not moving. After 2008 turns, determine the probability that the king is on a square with both coordinates even. An exact answer is required.

10 Determine the number of 8-tuples of nonnegative integers $(a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4)$ satisfying $0 \le a_k \le k$, for each $k=1,2,3,4$, and $a_1+a_2+a_3+a_4+2b_1+3b_2+4b_3+5b_4=19$.

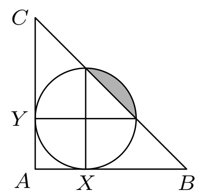
– Geometry

– February 23rd

1 How many different values can $\angle ABC$ take, where A, B, C are distinct vertices of a cube?

2 Let ABC be an equilateral triangle. Let Ω be its incircle (circle inscribed in the triangle) and let ω be a circle tangent externally to Ω as well as to sides AB and AC. Determine the ratio of the radius of Ω to the radius of ω.

3 Let ABC be a triangle with $\angle BAC = 90^\circ$. A circle is tangent to the sides AB and AC at X and Y respectively, such that the points on the circle diametrically opposite X and Y both lie on the side BC. Given that $AB = 6$, find the area of the portion of the circle that lies outside the triangle.



4 In a triangle ABC, take point D on BC such that $DB = 14, DA = 13, DC = 4$, and the circumcircle of ADB is congruent to the circumcircle of ADC. What is the area of triangle ABC?

5 A piece of paper is folded in half. A second fold is made at an angle ϕ ($0^\circ < \phi < 90^\circ$) to the first, and a cut is made as shown below.
12881
When the piece of paper is unfolded, the resulting hole is a polygon. Let O be one of its vertices. Suppose that all the other vertices of the hole lie on a circle centered at O, and also that $\angle XOY = 144^\circ$, where X and Y are the the vertices of the hole adjacent to O. Find the value(s) of ϕ (in degrees).

6 Let ABC be a triangle with $\angle A = 45^\circ$. Let P be a point on side BC with $PB = 3$ and $PC = 5$. Let O be the circumcenter of ABC. Determine the length OP.

7 Let C_1 and C_2 be externally tangent circles with radius 2 and 3, respectively. Let C_3 be a circle internally tangent to both C_1 and C_2 at points A and B, respectively. The tangents to C_3 at A and B meet at T, and $TA = 4$. Determine the radius of C_3.

8 Let ABC be an equilateral triangle with side length 2, and let Γ be a circle with radius $\frac{1}{2}$ centered at the center of the equilateral triangle. Determine the length of the shortest path that starts somewhere on Γ, visits all three sides of ABC, and ends somewhere on Γ (not necessarily at the starting point). Express your answer in the form of $\sqrt{p} - q$, where p and q are rational numbers written as reduced fractions.

9 Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D, and let lines BI and CI meet the circle with diameter AI at points P and Q, respectively. Given $BI = 6, CI = 5, DI = 3$, determine the value of $(DP/DQ)^2$.

10 Let ABC be a triangle with $BC = 2007$, $CA = 2008$, $AB = 2009$. Let ω be an excircle of ABC that touches the line segment BC at D, and touches extensions of lines AC and AB at E and F, respectively (so that C lies on segment AE and B lies on segment AF). Let O be the center of ω. Let ℓ be the line through O perpendicular to AD. Let ℓ meet line EF at G. Compute the length DG.

– General 1

– February 23rd

1 Let $ABCD$ be a unit square (that is, the labels A, B, C, D appear in that order around the square). Let X be a point outside of the square such that the distance from X to AC is equal to the distance from X to BD, and also that $AX = \frac{\sqrt{2}}{2}$. Determine the value of CX^2.

2 Find the smallest positive integer n such that $107n$ has the same last two digits as n.

3 There are 5 dogs, 4 cats, and 7 bowls of milk at an animal gathering. Dogs and cats are distinguishable, but all bowls of milk are the same. In how many ways can every dog and cat be paired with either a member of the other species or a bowl of milk such that all the bowls of milk are taken?

4 Positive real numbers x, y satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy.

5 The function f satisfies

$$f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1$$

for all real numbers x, y. Determine the value of $f(10)$.

6 In a triangle ABC, take point D on BC such that $DB = 14, DA = 13, DC = 4$, and the circumcircle of ADB is congruent to the circumcircle of ADC. What is the area of triangle ABC?

7 The equation $x^3 - 9x^2 + 8x + 2 = 0$ has three real roots p, q, r. Find $\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}$.

8 Let S be the smallest subset of the integers with the property that $0 \in S$ and for any $x \in S$, we have $3x \in S$ and $3x + 1 \in S$. Determine the number of non-negative integers in S less than 2008.

9 A Sudoku matrix is dened as a 9×9 array with entries from $\{1, 2, \ldots, 9\}$ and with the constraint that each row, each column, and each of the nine 3×3 boxes that tile the array contains each digit from 1 to 9 exactly once. A Sudoku matrix is chosen at random (so that every Sudoku matrix has equal probability of being chosen). We know two of the squares in this matrix, as shown. What is the probability that the square marked by ? contains the digit 3?

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– General 2

– February 23rd

1 Four students from Harvard, one of them named Jack, and five students from MIT, one of them named Jill, are going to see a Boston Celtics game. However, they found out that only 5 tickets remain, so 4 of them must go back. Suppose that at least one student from each school must go see the game, and at least one of Jack and Jill must go see the game, how many ways are there of choosing which 5 people can see the game?

2 Let ABC be an equilateral triangle. Let Ω be its incircle (circle inscribed in the triangle) and let ω be a circle tangent externally to Ω as well as to sides AB and AC. Determine the ratio of the radius of Ω to the radius of ω.

3 A $3\times3\times3$ cube composed of 27 unit cubes rests on a horizontal plane. Determine the number of ways of selecting two distinct unit cubes from a $3\times3\times1$ block (the order is irrelevant) with the property that the line joining the centers of the two cubes makes a 45° angle with the horizontal plane.

4 Suppose that a,b,c,d are real numbers satisfying $a\geq b\geq c\geq d\geq 0$, $a^2+d^2=1$, $b^2+c^2=1$, and $ac+bd=1/3$. Find the value of $ab-cd$.

5 Kermit the frog enjoys hopping around the innite square grid in his backyard. It takes him 1 Joule of energy to hop one step north or one step south, and 1 Joule of energy to hop one step east or one step west. He wakes up one morning on the grid with 100 Joules of energy, and hops till he falls asleep with 0 energy. How many different places could he have gone to sleep?

6 Determine all real numbers a such that the inequality $|x^2+2ax+3a|\leq 2$ has exactly one solution in x.

7 A *root of unity* is a complex number that is a solution to $z^n=1$ for some positive integer n. Determine the number of roots of unity that are also roots of $z^2+az+b=0$ for some integers a and b.

8 A piece of paper is folded in half. A second fold is made at an angle ϕ ($0^\circ<\phi<90^\circ$) to the first, and a cut is made as shown below.
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9 Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D, and let lines BI and CI meet the circle with diameter AI at points P and Q, respectively. Given $BI = 6, CI = 5, DI = 3$, determine the value of $(DP/DQ)^2$.

10 Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \leq a_k \leq k$, for each $k = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$.

– Guts

– February 23rd

1 Determine all pairs (a, b) of real numbers such that $10, a, b, ab$ is an arithmetic progression.

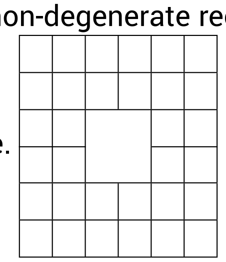
2 Given right triangle ABC, with $AB = 4, BC = 3$, and $CA = 5$. Circle ω passes through A and is tangent to BC at C. What is the radius of ω?

3 How many ways can you color the squares of a 2×2008 grid in 3 colors such that no two squares of the same color share an edge?

4 Find the real solution(s) to the equation $(x + y)^2 = (x + 1)(y - 1)$.

5 A Vandal and a Moderator are editing a Wikipedia article. The article originally is error-free. Each day, the Vandal introduces one new error into the Wikipedia article. At the end of the day, the moderator checks the article and has a $2/3$ chance of catching each individual error still in the article. After 3 days, what is the probability that the article is error-free?

6 Determine the number of non-degenerate rectangles whose edges lie completely on the grid lines of the following figure.



7 Given that $x + \sin y = 2008$ and $x + 2008 \cos y = 2007$, where $0 \leq y \leq \pi/2$, find the value of $x + y$.

8 Trodgor the dragon is burning down a village consisting of 90 cottages. At time $t = 0$ an angry peasant arises from each cottage, and every 8 minutes (480 seconds) thereafter another angry peasant spontaneously generates from each non-burned cottage. It takes Trodgor 5 seconds to either burn a peasant or to burn a cottage, but Trodgor cannot begin burning cottages until all the peasants around him have been burned. How many **seconds** does it take Trodgor to burn down the entire village?

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9 Consider a circular cone with vertex V, and let ABC be a triangle inscribed in the base of the cone, such that AB is a diameter and $AC = BC$. Let L be a point on BV such that the volume of the cone is 4 times the volume of the tetrahedron $ABCL$. Find the value of BL/LV.

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10 Find the number of subsets S of $\{1, 2, \ldots 63\}$ the sum of whose elements is 2008.

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11 Let $f(r) = \sum_{j=2}^{2008} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2008^r}$. Find $\sum_{k=2}^{\infty} f(k)$.

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12 Suppose we have an (infinite) cone $\mathcal{C}$ with apex A and a plane π. The intersection of π and $\mathcal{C}$ is an ellipse $\mathcal{E}$ with major axis BC, such that B is closer to A than C, and $BC = 4$, $AC = 5$, $AB = 3$. Suppose we inscribe a sphere in each part of $\mathcal{C}$ cut up by $\mathcal{E}$ with both spheres tangent to $\mathcal{E}$. What is the ratio of the radii of the spheres (smaller to larger)?

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13 Let $P(x)$ be a polynomial with degree 2008 and leading coefficient 1 such that

$$P(0) = 2007, P(1) = 2006, P(2) = 2005, \ldots, P(2007) = 0.$$

Determine the value of $P(2008)$. You may use factorials in your answer.

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14 Evaluate the infinite sum $\sum_{n=1}^{\infty} \frac{n}{n^4+4}$.

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15 In a game show, Bob is faced with 7 doors, 2 of which hide prizes. After he chooses a door, the host opens three other doors, of which one is hiding a prize. Bob chooses to switch to another door. What is the probability that his new door is hiding a prize?

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16 Point A lies at $(0, 4)$ and point B lies at $(3, 8)$. Find the x-coordinate of the point X on the x-axis maximizing $\angle AXB$.

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17 Solve the equation

$$\sqrt{x + \sqrt{4x + \sqrt{16x + \sqrt{\ldots + \sqrt{4^{2008}x + 3}}}}} - \sqrt{x} = 1.$$

Express your answer as a reduced fraction with the numerator and denominator written in their prime factorization.

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18 Let ABC be a right triangle with $\angle A = 90^\circ$. Let D be the midpoint of AB and let E be a point on segment AC such that $AD = AE$. Let BE meet CD at F. If $\angle BFC = 135^\circ$, determine BC/AB.

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19 Let $ABCD$ be a regular tetrahedron, and let O be the centroid of triangle BCD. Consider the point P on AO such that P minimizes $PA + 2(PB + PC + PD)$. Find $\sin \angle PBO$.

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20 For how many ordered triples (a, b, c) of positive integers are the equations $abc+9 = ab+bc+ca$ and $a + b + c = 10$ satisfied?

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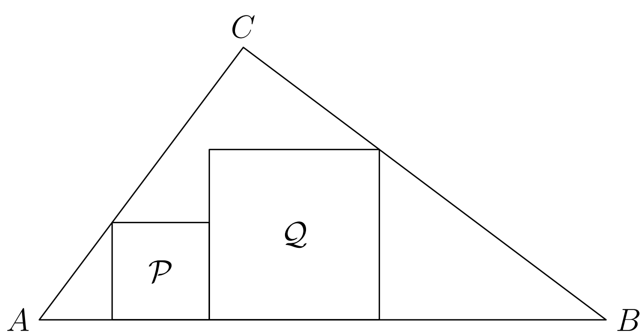
21 Let ABC be a triangle with $AB = 5$, $BC = 4$ and $AC = 3$. Let $\mathcal{P}$ and $\mathcal{Q}$ be squares inside ABC with disjoint interiors such that they both have one side lying on AB. Also, the two squares each have an edge lying on a common line perpendicular to AB, and $\mathcal{P}$ has one vertex on AC and $\mathcal{Q}$ has one vertex on BC. Determine the minimum value of the sum of the areas of the two squares.



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22 For a positive integer n, let $\theta(n)$ denote the number of integers $0 \le x < 2010$ such that $x^2 - n$ is divisible by 2010. Determine the remainder when $\displaystyle\sum_{n=0}^{2009} n \cdot \theta(n)$ is divided by 2010.

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23 Two mathematicians, Kelly and Jason, play a cooperative game. The computer selects some secret positive integer $n < 60$ (both Kelly and Jason know that $n < 60$, but that they don't know what the value of n is). The computer tells Kelly the unit digit of n, and it tells Jason the number of divisors of n. Then, Kelly and Jason have the following dialogue:

Kelly: I don't know what n is, and I'm sure that you don't know either. However, I know that n is divisible by at least two different primes.

Jason: Oh, then I know what the value of n is.

Kelly: Now I also know what n is.

Assuming that both Kelly and Jason speak truthfully and to the best of their knowledge, what are all the possible values of n?

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24 Suppose that ABC is an isosceles triangle with $AB = AC$. Let P be the point on side AC so that $AP = 2CP$. Given that $BP = 1$, determine the maximum possible area of ABC.

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25 Alice and the Cheshire Cat play a game. At each step, Alice either (1) gives the cat a penny, which causes the cat to change the number of (magic) beans that Alice has from n to $5n$ or (2) gives the cat a nickel, which causes the cat to give Alice another bean. Alice wins (and the cat disappears) as soon as the number of beans Alice has is greater than 2008 and has last two digits 42. What is the minimum number of cents Alice can spend to win the game, assuming she starts with 0 beans?

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26 Let $\mathcal{P}$ be a parabola, and let V_1 and F_1 be its vertex and focus, respectively. Let A and B be points on $\mathcal{P}$ so that $\angle AV_1B = 90^\circ$. Let $\mathcal{Q}$ be the locus of the midpoint of AB. It turns out that $\mathcal{Q}$ is also a parabola, and let V_2 and F_2 denote its vertex and focus, respectively. Determine the ratio F_1F_2/V_1V_2.

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27 Cyclic pentagon $ABCDE$ has a right angle $\angle ABC = 90^\circ$ and side lengths $AB = 15$ and $BC = 20$. Supposing that $AB = DE = EA$, find CD.

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28 Let P be a polyhedron where every face is a regular polygon, and every edge has length 1. Each vertex of P is incident to two regular hexagons and one square. Choose a vertex V of the polyhedron. Find the volume of the set of all points contained in P that are closer to V than to any other vertex.

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29 Let (x, y) be a pair of real numbers satisfying

$$56x + 33y = \frac{-y}{x^2+y^2}, \qquad \text{and} \qquad 33x - 56y = \frac{x}{x^2+y^2}.$$

Determine the value of $|x| + |y|$.

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30 Triangle ABC obeys $AB = 2AC$ and $\angle BAC = 120^\circ$. Points P and Q lie on segment BC such that

$$AB^2 + BC \cdot CP = BC^2$$
$$3AC^2 + 2BC \cdot CQ = BC^2$$

Find $\angle PAQ$ in degrees.

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31 Let $\mathcal{C}$ be the hyperbola $y^2 - x^2 = 1$. Given a point P_0 on the x-axis, we construct a sequence of points (P_n) on the x-axis in the following manner: let ℓ_n be the line with slope 1 passing passing through P_n, then P_{n+1} is the orthogonal projection of the point of intersection of ℓ_n and $\mathcal{C}$ onto the x-axis. (If $P_n = 0$, then the sequence simply terminates.)

Let N be the number of starting positions P_0 on the x-axis such that $P_0 = P_{2008}$. Determine the remainder of N when divided by 2008.

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32 Cyclic pentagon $ABCDE$ has side lengths $AB = BC = 5$, $CD = DE = 12$, and $AE = 14$. Determine the radius of its circumcircle.

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33 Let a, b, c be nonzero real numbers such that $a + b + c = 0$ and $a^3 + b^3 + c^3 = a^5 + b^5 + c^5$. Find the value of $a^2 + b^2 + c^2$.
