Harvard-MIT Mathematics Tournament 2009

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–	Algebra
1	If a and b are positive integers such that $a^2 - b^4 = 2009$, find $a + b$.
2	Let S be the sum of all the real coefficients of the expansion of $(1 + ix)^{2009}$. What is $\log_2(S)$?
3	If $\tan x + \tan y = 4$ and $\cot x + \cot y = 5$, compute $\tan(x + y)$.
4	Suppose a, b and c are integers such that the greatest common divisor of $x^2 + ax + b$ and $x^2 + bx + c$ is $x + 1$ (in the set of polynomials in x with integer coefficients), and the least common multiple of $x^2 + ax + b$ and $x^2 + bx + c$ $x^3 - 4x^2 + x + 6$. Find $a + b + c$.
5	Let a, b, and c be the 3 roots of $x^3 - x + 1 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.
6	Let x and y be positive real numbers and θ an angle such that $\theta \neq \frac{\pi}{2}n$ for any integer n. Suppose $$\frac{\sin\theta}{x} = \frac{\cos\theta}{y}$$ and $$\frac{\cos^4\theta}{x^4} + \frac{\sin^4\theta}{y^4} = \frac{97\sin 2\theta}{x^3y + y^3x}.$$ Compute $\frac{x}{y} + \frac{y}{x}$.
7	Simplify the product $$\prod_{m=1}^{100}\prod_{n=1}^{100}\frac{x^{n+m} + x^{n+m+2} + x^{2n+1} + x^{2m+1}}{x^{2n} + 2x^{n+m} + x^{2m}}$$ Express your answer in terms of x.
8	If a, b, x and y are real numbers such that $ax + by = 3$, $ax^2 + by^2 = 7$, $ax^3 + bx^3 = 16$, and $ax^4 + by^4 = 42$, find $ax^5 + by^5$.
9	Let $f(x) = x^4 + 14x^3 + 52x^2 + 56x + 16$. Let z_1, z_2, z_3, z_4 be the four roots of f. Find the smallest possible value of $\lvert z_a z_b + z_c z_d\rvert$ where $\{a, b, c, d\} = \{1, 2, 3, 4\}$.

10 Let $f(x) = 2x^3 - 2x$. For what positive values of a do there exist distinct b, c, d such that $(a, f(a)), (b, f(b)), (c, f(c)), (d, f(d))$ is a rectangle?

– Calculus

1 Let f be a diff erentiable real-valued function defi ned on the positive real numbers. The tangent lines to the graph of f always meet the y-axis 1 unit lower than where they meet the function. If $f(1) = 0$, what is $f(2)$?

2 The differentiable function $F : \mathbb{R} \to \mathbb{R}$ satisfies $F(0) = -1$ and

$$\frac{d}{dx}F(x) = \sin(\sin(\sin(\sin(x)))) \cdot \cos(\sin(\sin(x))) \cdot \cos(\sin(x)) \cdot \cos(x).$$

Find $F(x)$ as a function of x.

3 Compute e^A where A is defined as

$$\int_{3/4}^{4/3} \frac{2x^2 + x + 1}{x^3 + x^2 + x + 1} dx.$$

4 Let P be a fourth degree polynomial, with derivative P', such that $P(1) = P(3) = P(5) = P'(7) = 0$. Find the real number $x \neq 1, 3, 5$ such that $P(x) = 0$.

5 Compute

$$\lim_{h \to 0} \frac{\sin(\frac{\pi}{3} + 4h) - 4\sin(\frac{\pi}{3} + 3h) + 6\sin(\frac{\pi}{3} + 2h) - 4\sin(\frac{\pi}{3} + h) + \sin(\frac{\pi}{3})}{h^4}.$$

6 Let $p_0(x), p_1(x), p_2(x), \ldots$ be polynomials such that $p_0(x) = x$ and for all positive integers n, $\frac{d}{dx}p_n(x) = p_{n-1}(x)$. Define the function $p(x) : [0, \infty) \to \mathbb{R}$ by $p(x) = p_n(x)$ for all $x \in [n, n+1)$. Given that $p(x)$ is continuous on $[0, \infty)$, compute

$$\sum_{n=0}^{\infty} p_n(2009).$$

7 A line in the plane is called *strange* if it passes through $(a, 0)$ and $(0, 10 - a)$ for some a in the interval $[0, 10]$. A point in the plane is called *charming* if it lies in the first quadrant and also lies **below** some strange line. What is the area of the set of all charming points?

8 Compute

$$\int_1^{\sqrt{3}} x^{2x^2+1} + \ln\left(x^{2x^{2x^2+1}}\right) dx.$$

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9 Let $\mathcal{R}$ be the region in the plane bounded by the graphs of $y = x$ and $y = x^2$. Compute the volume of the region formed by revolving $\mathcal{R}$ around the line $y = x$.

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10 Let a and b be real numbers satisfying $a > b > 0$. Evaluate

$$\int_0^{2\pi} \frac{1}{a + b\cos(\theta)} d\theta.$$

Express your answer in terms of a and b.

– Combinatorics

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1 How many ways can the integers from -7 to 7 inclusive be arranged in a sequence such that the absolute value of the numbers in the sequence does not decrease?

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2 Two jokers are added to a 52 card deck and the entire stack of 54 cards is shuffled randomly. What is the expected number of cards that will be strictly between the two jokers?

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3 How many rearrangements of the letters of "$HMMTHMMT$" do not contain the substring "$HMMT$"? (For instance, one such arrangement is $HMMHMTMT$.)

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4 How many functions $f : f\{1,2,3,4,5\} \longrightarrow \{1,2,3,4,5\}$ satisfy $f(f(x)) = f(x)$ for all $x \in \{1,2,3,4,5\}$?

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5 Let $s(n)$ denote the number of 1's in the binary representation of n. Compute

$$\frac{1}{255} \sum_{0 \leq n < 16} 2^n (-1)^{s(n)}.$$

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6 How many sequences of 5 positive integers (a,b,c,d,e) satisfy $abcde \leq a+b+c+d+e \leq 10$?

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7 Paul fi lls in a 7×7 grid with the numbers 1 through 49 in a random arrangement. He then erases his work and does the same thing again, to obtain two diff erent random arrangements of the numbers in the grid. What is the expected number of pairs of numbers that occur in either the same row as each other or the same column as each other in both of the two arrangements?

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8 There are 5 students on a team for a math competition. The math competition has 5 subject tests. Each student on the team must choose 2 distinct tests, and each test must be taken by exactly two people. In how many ways can this be done?

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9 The squares of a 3×3 grid are filled with positive integers such that 1 is the label of the upper-leftmost square, 2009 is the label of the lower-rightmost square, and the label of each square divides the ne directly to the right of it and the one directly below it. How many such labelings are possible?

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10 Given a rearrangement of the numbers from 1 to n, each pair of consecutive elements a and b of the sequence can be either increasing (if $a < b$) or decreasing (if $b < a$). How many rearrangements of the numbers from 1 to n have exactly two increasing pairs of consecutive elements? Express your answer in terms of n.

– General Part 1

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1 If a and b are positive integers such that $a^2 - b^4 = 2009$, find $a + b$.

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2 Suppose N is a 6-digit number having base-10 representation $\underline{a}\ \underline{b}\ \underline{c}\ \underline{d}\ \underline{e}\ \underline{f}$. If N is $6/7$ of the number having base-10 representation $\underline{d}\ \underline{e}\ \underline{f}\ \underline{a}\ \underline{b}\ \underline{c}$, find N.

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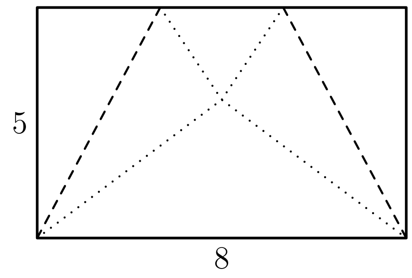
3 A rectangular piece of paper with side lengths 5 by 8 is folded along the dashed lines shown below, so that the folded flaps just touch at the corners as shown by the dotted lines. Find the area of the resulting trapezoid.



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4 If $\tan x + \tan y = 4$ and $\cot x + \cot y = 5$, compute $\tan(x + y)$.

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5 Two jokers are added to a 52 card deck and the entire stack of 54 cards is shuffled randomly. What is the expected number of cards that will be strictly between the two jokers?

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6 The corner of a unit cube is chopped off such that the cut runs through the three vertices adjacent to the vertex of the chosen corner. What is the height of the cube when the freshly-cut face is placed on a table?

7 Let $s(n)$ denote the number of 1's in the binary representation of n. Compute

$$\frac{1}{255}\sum_{0\le n<16} 2^n(-1)^{s(n)}.$$

8 Let a, b, and c be the 3 roots of $x^3 - x + 1 = 0$. Find $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}$.

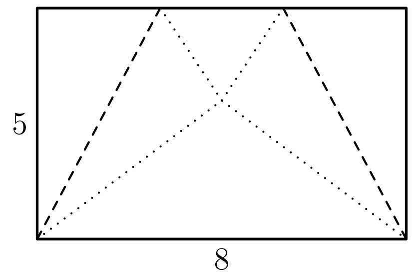
9 How many functions $f : f\{1,2,3,4,5\} \longrightarrow \{1,2,3,4,5\}$ satisfy $f(f(x)) = f(x)$ for all $x \in \{1,2,3,4,5\}$?

10 A *kite* is a quadrilateral whose diagonals are perpendicular. Let kite $ABCD$ be such that $\angle B = \angle D = 90^\circ$. Let M and N be the points of tangency of the incircle of $ABCD$ to AB and BC respectively. Let ω be the circle centered at C and tangent to AB and AD. Construct another kite $AB'C'D'$ that is similar to $ABCD$ and whose incircle is ω. Let N' be the point of tangency of $B'C'$ to ω. If $MN' \parallel AC$, then what is the ratio of $AB : BC$?

– General Part 2

– Geometry

1 A rectangular piece of paper with side lengths 5 by 8 is folded along the dashed lines shown below, so that the folded flaps just touch at the corners as shown by the dotted lines. Find the area of the resulting trapezoid.



2 The corner of a unit cube is chopped off such that the cut runs through the three vertices adjacent to the vertex of the chosen corner. What is the height of the cube when the freshly-cut face is placed on a table?

3 Let T be a right triangle with sides having lengths 3, 4, and 5. A point P is called *awesome* if P is the center of a parallelogram whose vertices all lie on the boundary of T. What is the area of the set of awesome points?

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4 A *kite* is a quadrilateral whose diagonals are perpendicular. Let kite $ABCD$ be such that $\angle B = \angle D = 90^\circ$. Let M and N be the points of tangency of the incircle of $ABCD$ to AB and BC respectively. Let ω be the circle centered at C and tangent to AB and AD. Construct another kite $AB'C'D'$ that is similar to $ABCD$ and whose incircle is ω. Let N' be the point of tangency of $B'C'$ to ω. If $MN' \parallel AC$, then what is the ratio of $AB : BC$?

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5 Circle B has radius $6\sqrt{7}$. Circle A, centered at point C, has radius $\sqrt{7}$ and is contained in B. Let L be the locus of centers C such that there exists a point D on the boundary of B with the following property: if the tangents from D to circle A intersect circle B again at X and Y, then XY is also tangent to A. Find the area contained by the boundary of L.

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6 Let ABC be a triangle in the coordinate plane with vertices on lattice points and with $AB = 1$. Suppose the perimeter of ABC is less than 17. Find the largest possible value of $1/r$, where r is the inradius of ABC.

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7 In triangle ABC, D is the midpoint of BC, E is the foot of the perpendicular from A to BC, and F is the foot of the perpendicular from D to AC. Given that $BE = 5$, $EC = 9$, and the area of triangle ABC is 84, compute $|EF|$.

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8 Triangle ABC has side lengths $AB = 231$, $BC = 160$, and $AC = 281$. Point D is constructed on the opposite side of line AC as point B such that $AD = 178$ and $CD = 153$. Compute the distance from B to the midpoint of segment AD.

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9 Let ABC be a triangle with $AB = 16$ and $AC = 5$. Suppose that the bisectors of angle $\angle ABC$ and $\angle BCA$ meet at a point P in the triangle's interior. Given that $AP = 4$, compute BC.

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10 Points A and B lie on circle ω. Point P lies on the extension of segment AB past B. Line ℓ passes through P and is tangent to ω. The tangents to ω at points A and B intersect ℓ at points D and C respectively. Given that $AB = 7$, $BC = 2$, and $AD = 3$, compute BP.

– Guts

– Team A

– Team B
