Harvard-MIT Mathematics Tournament 2010

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by djmathman

–	Algebra
1	Suppose that x and y are positive reals such that $$x - y^2 = 3, \qquad x^2 + y^4 = 13.$$ Find x.
2	The *rank* of a rational number q is the unique k for which $q = \frac{1}{a_1} + \cdots + \frac{1}{a_k}$, where each a_i is the smallest positive integer q such that $q \geq \frac{1}{a_1} + \cdots + \frac{1}{a_i}$. Let q be the largest rational number less than $\frac{1}{4}$ with rank 3, and suppose the expression for q is $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$. Find the ordered triple (a_1, a_2, a_3).
3	Let $S_0 = 0$ and let S_k equal $a_1 + 2a_2 + \ldots + ka_k$ for $k \geq 1$. Define a_i to be 1 if $S_{i-1} < i$ and -1 if $S_{i-1} \geq i$. What is the largest $k \leq 2010$ such that $S_k = 0$?
4	Suppose that there exist nonzero complex numbers a, b, c, and d such that k is a root of both the equations $ax^3 + bx^2 + cx + d = 0$ and $bx^3 + cx^2 + dx + a = 0$. Find all possible values of k (including complex values).
5	Suppose that x and y are complex numbers such that $x + y = 1$ and $x^{20} + y^{20} = 20$. Find the sum of all possible values of $x^2 + y^2$.
6	Suppose that a polynomial of the form $p(x) = x^{2010} \pm x^{2009} \pm \cdots \pm x \pm 1$ has no real roots. What is the maximum possible number of coefficients of -1 in p?
7	Let $a, b, c, x, y,$ and z be complex numbers such that $$a = \frac{b+c}{x-2}, \qquad b = \frac{c+a}{y-2}, \qquad c = \frac{a+b}{z-2}.$$ If $xy + yz + xz = 67$ and $x + y + z = 2010$, find the value of xyz.
8	How many polynomials of degree exactly 5 with real coefficients send the set $\{1, 2, 3, 4, 5, 6\}$ to a permutation of itself?
9	Let $f(x) = cx(x-1)$, where c is a positive real number. We use $f^n(x)$ to denote the polynomial obtained by composing f with itself n times. For every positive integer n, all the roots of $f^n(x)$ are real. What is the smallest possible value of c?

10 Let $p(x)$ and $q(x)$ be two cubic polynomials such that $p(0) = -24$, $q(0) = 30$, and

$$p(q(x)) = q(p(x))$$

for all real numbers x. Find the ordered pair $(p(3), q(6))$.

– Calculus

1 Suppose that $p(x)$ is a polynomial and that $p(x) - p'(x) = x^2 + 2x + 1$. Compute $p(5)$.

2 Let f be a function such that $f(0) = 1$, $f'(0) = 2$, and

$$f''(t) = 4f'(t) - 3f(t) + 1$$

for all t. Compute the 4th derivative of f, evaluated at 0.

3 Let p be a monic cubic polynomial such that $p(0) = 1$ and such that all the zeroes of $p'(x)$ are also zeroes of $p(x)$. Find p. Note: monic means that the leading coefficient is 1.

4 Compute $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} |\cos(k)|}{n}$.

5 Let the functions $f(\alpha, x)$ and $g(\alpha)$ be defined as

$$f(\alpha, x) = \frac{(\frac{x}{2})^\alpha}{x-1} \qquad g(\alpha) = \frac{d^4 f}{dx^4}|_{x=2}$$

Then $g(\alpha)$ is a polynomial is α. Find the leading coefficient of $g(\alpha)$.

6 Let $f(x) = x^3 - x^2$. For a given value of x, the graph of $f(x)$, together with the graph of the line $c + x$, split the plane up into regions. Suppose that c is such that exactly two of these regions have finite area. Find the value of c that minimizes the sum of the areas of these two regions.

7 Let a_1, a_2, and a_3 be nonzero complex numbers with non-negative real and imaginary parts. Find the minimum possible value of

$$\frac{|a_1 + a_2 + a_3|}{\sqrt[3]{|a_1 a_2 a_3|}}.$$

8 Let $f(n) = \sum_{k=2}^{\infty} \frac{1}{k^n \cdot k!}$. Calculate $\sum_{n=2}^{\infty} f(n)$.

9 Let $x(t)$ be a solution to the differential equation

$$(x+x')^2 + x \cdot x'' = \cos t$$

with $x(0) = x'(0) = \sqrt{\frac{2}{5}}$. Compute $x\left(\frac{\pi}{4}\right)$.

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10 Let $f(n) = \sum_{k=1}^{n} \frac{1}{k}$. Then there exists constants γ, c, and d such that

$$f(n) = \ln(x) + \gamma + \frac{c}{n} + \frac{d}{n^2} + O\left(\frac{1}{n^3}\right),$$

where the $O\left(\frac{1}{n^3}\right)$ means terms of order $\frac{1}{n^3}$ or lower. Compute the ordered pair (c, d).

– Combinatorics

– General Part 1

– General Part 2

– Geometry

– Guts

– Team A

– Team B