Harvard-MIT Mathematics Tournament 2011

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by orangefronted, v_Enhance, ProblemSolver1026, dragon96, Geometrizer

– Algebra

– February 12th

1 Let a, b, c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: ax^2+bx+c, bx^2+cx+a, and cx^2+ax+b.

2 Let $a \star b = ab + a + b$ for all integers a and b. Evaluate $1 \star (2 \star (3 \star (4 \star \ldots (99 \star 100) \ldots)))$.

3 Find all integers x such that $2x^2+x-6$ is a positive integral power of a prime positive integer.

4 For all real numbers x, let

$$f(x) = \frac{1}{\sqrt[2011]{1-x^{2011}}}.$$

Evaluate $(f(f(\ldots(f(2011))\ldots)))^{2011}$, where f is applied 2010 times.

5 Let $f(x) = x^2 + 6x + c$ for all real number sx, where c is some real number. For what values of c does $f(f(x))$ have exactly 3 distinct real roots?

6 How many polynomials P with integer coefficients and degree at most 5 satisfy $0 \leq P(x) < 120$ for all $x \in \{0,1,2,3,4,5\}$?

7 Let $A = \{1, 2, \ldots, 2011\}$. Find the number of functions f from A to A that satisfy $f(n) \leq n$ for all n in A and attain exactly 2010 distinct values.

8 Let $z = \cos\frac{2\pi}{2011} + i\sin\frac{2\pi}{2011}$, and let

$$P(x) = x^{2008} + 3x^{2007} + 6x^{2006} + \cdots + \frac{2008 \cdot 2009}{2}x + \frac{2009 \cdot 2010}{2}$$

for all complex numbers x. Evaluate $P(z)P(z^2)P(z^3)\cdots P(z^{2010})$.

9 Let $\{a_n\}$ and $\{b_n\}$ be sequences defined recursively by $a_0 = 2$; $b_0 = 2$, and $a_{n+1} = a_n\sqrt{1+a_n^2+b_n^2} - b_n$; $b_{n+1} = b_n\sqrt{1+a_n^2+b_n^2} + a_n$. Find the ternary (base 3) representation of a_4 and b_4.

– Geometry

– February 12th

1 Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D. If there exist points E and F on sides AC and BC, respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC.

2 Let H be a regular hexagon of side length x. Call a hexagon in the same plane a "distortion" of H if
and only if it can be obtained from H by translating each vertex of H by a distance strictly less than 1. Determine the smallest value of x for which every distortion of H is necessarily convex.

3 Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE. Let X be the intersection of AC and BM, let Y be the intersection of BF and AM, and let Z be the intersection of AC and BF. If $[P]$ denotes the area of a polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

4 Let $ABCD$ be a square of side length 13. Let E and F be points on rays AB and AD respectively, so that the area of square $ABCD$ equals the area of triangle AEF. If EF intersects BC at X and $BX = 6$, determine DF.

5 Let $ABCDEF$ be a convex equilateral hexagon such that lines BC, AD, and EF are parallel. Let H be the orthocenter of triangle ABD. If the smallest interior angle of the hexagon is 4 degrees, determine the smallest angle of the triangle HAD in degrees.

6 Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD. If $AI = 2$ as well, find the minimum value of the length of diagonal BD.

7 Let $ABCD$ be a quadrilateral inscribed in the unit circle such that $\angle BAD$ is 30 degrees. Let m denote the minimum value of $CP + PQ + CQ$, where P and Q may be any points lying along rays AB and AD, respectively. Determine the maximum value of m.

8 Collinear points A, B, and C are given in the Cartesian plane such that $A = (a, 0)$ lies along the x-axis, B lies along the line $y = x$, C lies along the line $y = 2x$, and $\frac{AB}{BC} = 2$. If $D = (a, a)$, and the circumcircle of triangle ADC intersects the line $y = x$ again at E, and ray AE intersects $y = 2x$ at F, evaluate $\frac{AE}{EF}$.

9 Let ω_1 and ω_2 be two circles that intersect at points A and B. Let line l be tangent to ω_1 at P and to ω_2 at Q such that A is closer to PQ than B. Let points R and S lie along rays PA and QA, respectively, so that $PQ = AR = AS$ and R and S are on opposite sides of A as P and

Q. Let O be the circumcenter of triangle ASR, and C and D be the midpoints of major arcs AP and AQ, respectively. If $\angle APQ$ is 45 degrees and $\angle AQP$ is 30 degrees, determine $\angle COD$ in degrees.

– Combinatorics

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2 A classroom has 30 students and 30 desks arranged in 5 rows of 6. If the class has 15 boys and 15 girls, in how many ways can the students be placed in the chairs such that no boy is sitting in front of, behind, or next to another boy, and no girl is sitting in front of, behind, or next to another girl?

3 Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a
running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else inde nitely. If Nathaniel goes fi rst, determine the probability that he ends up winning.

– Calculus

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3 Evaluate $\int_1^\infty \left(\frac{\ln x}{x}\right)^{2011} dx$.