Harvard-MIT Mathematics Tournament 2012

– **Algebra**

1 Let f be the function such that

$$f(x) = \begin{cases} 2x & \text{if } x \le \frac{1}{2} \\ 2 - 2x & \text{if } x > \frac{1}{2} \end{cases}$$

What is the total length of the graph of $\underbrace{f(f(\ldots f(x)\ldots))}_{2012\ f's}$ from $x = 0$ to $x = 1$?

2 You are given an unlimited supply of red, blue, and yellow cards to form a hand. Each card has a point value and your score is the sum of the point values of those cards. The point values are as follows: the value of each red card is 1, the value of each blue card is equal to twice the number of red cards, and the value of each yellow card is equal to three times the number of blue cards. What is the maximum score you can get with fteen cards?

3 Given points a and b in the plane, let $a \oplus b$ be the unique point c such that abc is an equilateral triangle with a, b, c in the clockwise orientation.
Solve $(x \oplus (0, 0)) \oplus (1, 1) = (1, -1)$ for x.

4 During the weekends, Eli delivers milk in the complex plane. On Saturday, he begins at z and delivers milk to houses located at $z^3, z^5, z^7, \ldots, z^{2013}$ in that order; on Sunday, he begins at 1 and delivers milk to houses located at $z^2, z^4, z^6, \ldots, z^{2012}$ in that order. Eli always walks directly (in a straight line) between two houses. If the distance he must travel from his starting point to the last house is $\sqrt{2012}$ on both days, find the real part of z^2.

5 Find all ordered triples (a, b, c) of positive reals that satisfy: $\lfloor a \rfloor bc = 3, a \lfloor b \rfloor c = 4$, and $ab \lfloor c \rfloor = 5$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

6 Let $a_0 = -2, b_0 = 1$, and for $n \ge 0$, let

$$a_{n+1} = a_n + b_n + \sqrt{a_n^2 + b_n^2},$$
$$b_{n+1} = a_n + b_n - \sqrt{a_n^2 + b_n^2}.$$

Find a_{2012}.

7   Let $\otimes$ be a binary operation that takes two positive real numbers and returns a positive real number. Suppose further that $\otimes$ is continuous, commutative ($a \otimes b = b \otimes a$), distributive across multiplication ($a \otimes (bc) = (a \otimes b)(a \otimes c)$), and that $2 \otimes 2 = 4$. Solve the equation $x \otimes y = x$ for y in terms of x for $x > 1$.

8   Let $x_1 = y_1 = x_2 = y_2 = 1$, then for $n \geq 3$ let $x_n = x_{n-1}y_{n-2} + x_{n-2}y_{n-1}$ and $y_n = y_{n-1}y_{n-2} - x_{n-1}x_{n-2}$. What are the last two digits of $|x_{2012}|$?

9   How many real triples (a, b, c) are there such that the polynomial $p(x) = x^4 + ax^3 + bx^2 + ax + c$ has exactly three distinct roots, which are equal to $\tan y$, $\tan 2y$, and $\tan 3y$ for some real number y?

10   Suppose that there are 16 variables $\{a_{i,j}\}_{0 \leq i,j \leq 3}$, each of which may be 0 or 1. For how many settings of the variables $a_{i,j}$ do there exist positive reals $c_{i,j}$ such that the polynomial

$$f(x, y) = \sum_{0 \leq i,j \leq 3} a_{i,j}c_{i,j}x^i y^j$$

$(x, y \in \mathbb{R})$ is bounded below?

–   Combinatorics

–   Geometry