Harvard-MIT Mathematics Tournament 2013

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– Algebra

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1 Let x and y be real numbers with $x > y$ such that $x^2y^2 + x^2 + y^2 + 2xy = 40$ and $xy + x + y = 8$. Find the value of x.

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2 Let $\{a_n\}_{n\geq 1}$ be an arithmetic sequence and $\{g_n\}_{n\geq 1}$ be a geometric sequence such that the first four terms of $\{a_n + g_n\}$ are 0, 0, 1, and 0, in that order. What is the 10th term of $\{a_n + g_n\}$?

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3 Let S be the set of integers of the form $2^x + 2^y + 2^z$, where x, y, z are pairwise distinct non-negative integers. Determine the 100th smallest element of S.

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4 Determine all real values of A for which there exist distinct complex numbers x_1, x_2 such that the following three equations hold:

$$x_1(x_1 + 1) = A$$
$$x_2(x_2 + 1) = A$$
$$x_1^4 + 3x_1^3 + 5x_1 = x_2^4 + 3x_2^3 + 5x_2.$$

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5 Let a and b be real numbers, and let r, s, and t be the roots of $f(x) = x^3 + ax^2 + bx - 1$. Also, $g(x) = x^3 + mx^2 + nx + p$ has roots r^2, s^2, and t^2. If $g(-1) = -5$, find the maximum possible value of b.

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6 Find the number of integers n such that

$$1 + \left\lfloor \frac{100n}{101} \right\rfloor = \left\lceil \frac{99n}{100} \right\rceil.$$

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7 Compute

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1 + a_2 + \cdots + a_7}{3^{a_1+a_2+\cdots+a_7}}.$$

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8 Let x, y be complex numbers such that $\dfrac{x^2 + y^2}{x + y} = 4$ and $\dfrac{x^4 + y^4}{x^3 + y^3} = 2$. Find all possible values of $\dfrac{x^6 + y^6}{x^5 + y^5}$.

9 Let z be a non-real complex number with $z^{23} = 1$. Compute

$$\sum_{k=0}^{22} \frac{1}{1 + z^k + z^{2k}}.$$

10 Let N be a positive integer whose decimal representation contains 11235 as a contiguous substring, and let k be a positive integer such that $10^k > N$. Find the minimum possible value of

$$\frac{10^k - 1}{\gcd(N, 10^k - 1)}.$$

– Combinatorics

– Geometry

– Team

1 Let a and b be real numbers such that $\frac{ab}{a^2+b^2} = \frac{1}{4}$. Find all possible values of $\frac{|a^2-b^2|}{a^2+b^2}$.

2 A cafe has 3 tables and 5 individual counter seats. People enter in groups of size between 1 and 4, inclusive, and groups never share a table. A group of more than 1 will always try to sit at a table, but will sit in counter seats if no tables are available. Conversely, a group of 1 will always try to sit at the counter rst. One morning, M groups consisting of a total of N people enter and sit down. Then, a single person walks in, and realizes that all the tables and counter seats are occupied by some person or group. What is the minimum possible value of $M + N$?

3 Let ABC be a triangle with circumcenter O such that $AC = 7$. Suppose that the circumcircle of AOC is tangent to BC at C and intersects the line AB at A and F. Let FO intersect BC at E. Compute BE.

4 Let a_1, a_2, a_3, a_4, a_5 be real numbers whose sum is 20. Determine with proof the smallest possible value of

$$\sum_{1 \le i \le j \le 5} \lfloor a_i + a_j \rfloor.$$

5 Thaddeus is given a 2013×2013 array of integers each between 1 and 2013, inclusive. He is allowed two operations:

1. Choose a row, and subtract 1 from each entry.

2. Chooses a column, and add 1 to each entry.

He would like to get an array where all integers are divisible by 2013. On how many arrays is this possible?

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6 Let triangle ABC satisfy $2BC = AB + AC$ and have incenter I and circumcircle ω. Let D be the intersection of AI and ω (with A, D distinct). Prove that I is the midpoint of AD.

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7 There are are n children and n toys such that each child has a strict preference ordering on the toys. We want to distribute the toys: say a distribution A dominates a distribution $B \neq A$ if in A, each child receives at least as preferable of a toy as in B. Prove that if some distribution is not dominated by any other, then at least one child gets his/her favorite toy in that distribution.

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8 Let points A and B be on circle ω centered at O. Suppose that ω_A and ω_B are circles not containing O which are internally tangent to ω at A and B, respectively. Let ω_A and ω_B intersect at C and D such that D is inside triangle ABC. Suppose that line BC meets ω again at E and let line EA intersect ω_A at F. If $FC \perp CD$, prove that O, C, and D are collinear.

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9 Let m be an odd positive integer greater than 1. Let S_m be the set of all non-negative integers less than m which are of the form $x + y$, where $xy - 1$ is divisible by m. Let $f(m)$ be the number of elements of S_m.

(a) Prove that $f(mn) = f(m)f(n)$ if m, n are relatively prime odd integers greater than 1.
(b) Find a closed form for $f(p^k)$, where $k > 0$ is an integer and p is an odd prime.

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10 Chim Tu has a large rectangular table. On it, there are nitely many pieces of paper with nonoverlapping interiors, each one in the shape of a convex polygon. At each step, Chim Tu is allowed to slide one piece of paper in a straight line such that its interior does not touch any other piece of paper during the slide. Can Chim Tu always slide all the pieces of paper off the table in nitely many steps?

– Guts

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1 Arpon chooses a positive real number k. For each positive integer n, he places a marker at the point (n, nk) in the (x, y) plane. Suppose that two markers whose x-coordinates differ by 4 have distance 31. What is the distance between the markers $(7, 7k)$ and $(19, 19k)$?

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2 The real numbers x, y, z, satisfy $0 \leq x \leq y \leq z \leq 4$. If their squares form an arithmetic progression with common difference 2, determine the minimum possible value of $|x-y|+|y-z|$.

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3 Find the rightmost non-zero digit of the expansion of $(20)(13!)$.

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4 Spencer is making burritos, each of which consists of one wrap and one filling. He has enough filling for up to four beef burritos and three chicken burritos. However, he only has five wraps for the burritos; in how many orders can he make exactly five burritos?

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5 Rahul has ten cards face-down, which consist of five distinct pairs of matching cards. During each move of his game, Rahul chooses one card to turn face-up, looks at it, and then chooses another to turn face-up and looks at it. If the two face-up cards match, the game ends. If not, Rahul flips both cards face-down and keeps repeating this process. Initially, Rahul doesn't know which cards are which. Assuming that he has perfect memory, find the smallest number of moves after which he can guarantee that the game has ended.

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6 Let R be the region in the Cartesian plane of points (x, y) satisfying $x \geq 0$, $y \geq 0$, and $x + y + \lfloor x \rfloor + \lfloor y \rfloor \leq 5$. Determine the area of R.

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7 Find the number of positive divisors d of $15! = 15 \cdot 14 \cdot \cdots \cdot 2 \cdot 1$ such that $\gcd(d, 60) = 5$.

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8 In a game, there are three indistinguishable boxes; one box contains two red balls, one contains two blue balls, and the last contains one ball of each color. To play, Raj first predicts whether he will draw two balls of the same color or two of different colors. Then, he picks a box, draws a ball at random,
looks at the color, and replaces the ball in the same box. Finally, he repeats this; however, the boxes are not shuffled between draws, so he can determine whether he wants to draw again from the same box. Raj wins if he predicts correctly; if he plays optimally, what is the probability that he will win?

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9 I have 8 unit cubes of different colors, which I want to glue together into a $2 \times 2 \times 2$ cube. How many distinct $2 \times 2 \times 2$ cubes can I make? Rotations of the same cube are not considered distinct, but reflections are.

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10 Wesyu is a farmer, and she's building a cao (a relative of the cow) pasture. Shw starts with a triangle $A_0A_1A_2$ where angle A_0 is 90°, angle A_1 is 60°, and A_0A_1 is 1. She then extends the pasture. FIrst, she extends A_2A_0 to A_3 such that $A_3A_0 = \frac{1}{2}A_2A_0$ and the new pasture is triangle $A_1A_2A_3$. Next, she extends A_3A_1 to A_4 such that $A_4A_1 = \frac{1}{6}A_3A_1$. She continues, each time extending A_nA_{n-2} to A_{n+1} such that $A_{n+1}A_{n-2} = \frac{1}{2^n - 2}A_nA_{n-2}$. What is the smallest K such that her pasture never exceeds an area of K?

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11 Compute the prime factorization of 1007021035035021007001. (You should write your answer in the form $p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$ where $p_1, \ldots, p_k$ are distinct prime numbers and $e_1, \ldots, e_k$ are positive integers.)

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12 For how many integers $1 \leq k \leq 2013$ does the decimal representation of k^k end with a 1?

13 Find the smallest positive integer n such that $\frac{5^{n+1}+2^{n+1}}{5^n+2^n} > 4.99$.

14 Consider triangle ABC with $\angle A = 2\angle B$. The angle bisectors from A and C intersect at D, and the angle bisector from C intersects $\overline{AB}$ at E. If $\frac{DE}{DC} = \frac{1}{3}$, compute $\frac{AB}{AC}$.

15 Tim and Allen are playing a match of *tenus*. In a match of *tenus*, the two players play a series of games, each of which is won by one of the two players. The match ends when one player has won exactly two more games than the other player, at which point the player who has won more games wins the match. In odd-numbered games, Tim wins with probability $3/4$, and in the even-numbered games, Allen wins with probability $3/4$. What is the expected number of games in a match?

16 The walls of a room are in the shape of a triangle ABC with $\angle ABC = 90^\circ$, $\angle BAC = 60^\circ$, and $AB = 6$. Chong stands at the midpoint of BC and rolls a ball toward AB. Suppose that the ball bounces off AB, then AC, then returns exactly to Chong. Find the length of the path of the ball.

17 The lines $y = x$, $y = 2x$, and $y = 3x$ are the three medians of a triangle with perimeter 1. Find the length of the longest side of the triangle.

18 Define the sequence of positive integers $\{a_n\}$ as follows. Let $a_1 = 1$, $a_2 = 3$, and for each $n > 2$, let a_n be the result of expressing a_{n-1} in base $n-1$, then reading the resulting numeral in base n, then adding 2 (in base n). For example, $a_2 = 3_{10} = 11_2$, so $a_3 = 11_3 + 2_3 = 6_{10}$. Express a_{2013} in base 10.

19 An isosceles trapezoid $ABCD$ with bases AB and CD has $AB = 13$, $CD = 17$, and height 3. Let E be the intersection of AC and BD. Circles Ω and ω are circumscribed about triangles ABE and CDE. Compute the sum of the radii of Ω and ω.

20 The polynomial $f(x) = x^3 - 3x^2 - 4x + 4$ has three real roots r_1, r_2, and r_3. Let $g(x) = x^3 + ax^2 + bx + c$ be the polynomial which has roots s_1, s_2, and s_3, where $s_1 = r_1 + r_2 z + r_3 z^2$, $s_2 = r_1 z + r_2 z^2 + r_3$, $s_3 = r_1 z^2 + r_2 + r_3 z$, and $z = \frac{-1+i\sqrt{3}}{2}$. Find the real part of the sum of the coefficients of $g(x)$.

21 Find the number of positive integers $j \leq 3^{2013}$ such that

$$j = \sum_{k=0}^{m} \left((-1)^k \cdot 3^{a_k}\right)$$

for some strictly increasing sequence of nonnegative integers $\{a_k\}$. For example, we may write $3 = 3^1$ and $55 = 3^0 - 3^3 + 3^4$, but 4 cannot be written in this form.

22 Sherry and Val are playing a game. Sherry has a deck containing 2011 red cards and 2012 black cards, shuffled randomly. Sherry flips these cards over one at a time, and before she flips each card over, Val guesses whether it is red or black. If Val guesses correctly, she wins 1 dollar; otherwise, she loses 1 dollar. In addition, Val must guess red exactly 2011 times. If Val plays optimally, what is her expected profit from this game?

23 Let $ABCD$ be a parallelogram with $AB = 8$, $AD = 11$, and $\angle BAD = 60^\circ$. Let X be on segment CD with $CX/XD = 1/3$ and Y be on segment AD with $AY/YD = 1/2$. Let Z be on segment AB such that AX, BY, and DZ are concurrent. Determine the area of triangle XYZ.

24 Given a point p and a line segment l, let $d(p, l)$ be the distance between them. Let A, B, and C be points in the plane such that $AB = 6$, $BC = 8$, $AC = 10$. What is the area of the region in the (x, y)-plane formed by the ordered pairs (x, y) such that there exists a point P inside triangle ABC with $d(P, AB) + x = d(P, BC) + y = d(P, AC)$?

25 The sequence (z_n) of complex numbers satisfies the following properties:

-z_1 and z_2 are not real.
-$z_{n+2} = z_{n+1}^2 z_n$ for all integers $n \geq 1$.
-$\frac{z_{n+3}}{z_n^2}$ is real for all integers $n \geq 1$.
-$\left|\frac{z_3}{z_4}\right| = \left|\frac{z_4}{z_5}\right| = 2$.
Find the product of all possible values of z_1.

26 Triangle ABC has perimeter 1. Its three altitudes form the side lengths of a triangle. Find the set of all possible values of $\min(AB, BC, CA)$.

27 Let W be the hypercube $\{(x_1, x_2, x_3, x_4) \,|\, 0 \leq x_1, x_2, x_3, x_4 \leq 1\}$. The intersection of W and a hyperplane parallel to $x_1 + x_2 + x_3 + x_4 = 0$ is a non-degenerate 3-dimensional polyhedron. What is the maximum number of faces of this polyhedron?

28 Let $z_0 + z_1 + z_2 + \cdots$ be an infinite complex geometric series such that $z_0 = 1$ and $z_{2013} = \frac{1}{2013^{2013}}$. Find the sum of all possible sums of this series.

29 Let $A_1, A_2, \ldots, A_m$ be finite sets of size 2012 and let $B_1, B_2, \ldots, B_m$ be finite sets of size 2013 such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Find the maximum value of m.

30 How many positive integers k are there such that

$$\frac{k}{2013}(a+b) = lcm(a, b)$$

has a solution in positive integers (a, b)?

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31 Let $ABCD$ be a quadrilateral inscribed in a unit circle with center O. Suppose that $\angle AOB = \angle COD = 135^\circ$, $BC = 1$. Let B' and C' be the reflections of A across BO and CO respectively. Let H_1 and H_2 be the orthocenters of $AB'C'$ and BCD, respectively. If M is the midpoint of OH_1, and O' is the reflection of O about the midpoint of MH_2, compute OO'.

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32 For an even positive integer n Kevin has a tape of length $4n$ with marks at $-2n, -2n+1, \ldots, 2n-1, 2n$. He then randomly picks n points in the set $-n, -n+1, -n+2, \ldots, n-1, n$ and places a stone on each of these points. We call a stone 'stuck' if it is on $2n$ or $-2n$, or either all the points to the right, or all the points to the left, all contain stones. Then, every minute, Kevin shifts the unstruck stones in the following manner:

-He picks an unstuck stone uniformly at random and then flips a fair coin.
-If the coin came up heads, he then moves that stone and every stone in the largest contiguous set containing that stone one point to the left. If the coin came up tails, he moves every stone in that set one point right instead.
-He repeats until all the stones are stuck.
Let p_n be the probability that at the end of the process there are exactly k stones in the right half. Evaluate

$$\frac{p_{n-1} - p_{n-2} + p_{n-3} + \ldots + p_3 - p_2 + p_1}{p_{n-1} + p_{n-2} + p_{n-3} + \ldots + p_3 + p_2 + p_1}$$

in terms of n.

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33 Compute the value of $1^{25} + 2^{24} + 3^{23} + \ldots + 24^2 + 25^1$. If your answer is A and the correct answer is C, then your score on this problem will be $\left\lfloor 25 \min\left(\left(\frac{A}{C}\right)^2, \left(\frac{C}{A}\right)^2\right)\right\rfloor$.

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34 For how many unordered sets $\{a, b, c, d\}$ of positive integers, none of which exceed 168, do there exist integers w, x, y, z such that $(-1)^w a + (-1)^x b + (-1)^y c + (-1)^z d = 168$? If your answer is A and the correct answer is C, then your score on this problem will be $\left\lfloor 25e^{-3\frac{|C-A|}{C}} \right\rfloor$.

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35 Let P be the number of ways to partition 2013 into an ordered tuple of prime numbers. What is $\log_2(P)$? If your answer is A and the correct answer is C, then your score on this problem will be $\left\lfloor \frac{125}{2}\left(\min\left(\frac{C}{A}, \frac{A}{C}\right) - \frac{3}{5}\right)\right\rfloor$ or zero, whichever is larger.

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36 (Mathematicians A to Z) Below are the names of 26 mathematicians, one for each letter of the alphabet. Your answer to this question should be a subset of $\{A, B, \cdots, Z\}$, where each letter represents the corresponding mathematician. If two mathematicians in your subset have birthdates that are within 20 years of each other, then your score is 0. Otherwise, your score is $\max(3(k-3), 0)$ where k is the number of elements in your set.

Niels Abel	Isaac Newton
Etienne Bezout	Nicole Oresme
Augustin-Louis Cauchy	Blaise Pascal
Rene Descartes	Daniel Quillen
Leonhard Euler	Bernhard Riemann
Pierre Fatou	Jean-Pierre Serre
Alexander Grothendieck	Alan Turing
David Hilbert	Stanislaw Ulam
Kenkichi Iwasawa	John Venn
Carl Jacobi	Andrew Wiles
Andrey Kolmogorov	Leonardo Ximenes
Joseph-Louis Lagrange	Shing-Tung Yau
John Milnor	Ernst Zermelo