Harvard-MIT Mathematics Tournament 2014

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– Algebra

1 Given that x and y are nonzero real numbers such that $x + \frac{1}{y} = 10$ and $y + \frac{1}{x} = \frac{5}{12}$, find all possible values of x.

2 Find the integer closest to

$$\frac{1}{\sqrt[4]{5^4+1} - \sqrt[4]{5^4-1}}$$

3 Let

$$A = \frac{1}{6}((\log_2(3))^3 - (\log_2(6))^3 - (\log_2(12))^3 + (\log_2(24))^3)$$

.

Compute 2^A.

4 Let b and c be real numbers and define the polynomial $P(x) = x^2 + bx + c$. Suppose that $P(P(1)) = P(P(2)) = 0$, and that $P(1) \neq P(2)$. Find $P(0)$.

5 Find the sum of all real numbers x such that $5x^4 - 10x^3 + 10x^2 - 5x - 11 = 0$.

6 Given w and z are complex numbers such that $|w+z| = 1$ and $|w^2+z^2| = 14$, find the smallest possible value of $|w^3+z^3|$. Here $|\cdot|$ denotes the absolute value of a complex number, given by $|a+bi| = \sqrt{a^2+b^2}$ whenever a and b are real numbers.

7 Find the largest real number c such that

$$\sum_{i=1}^{101} x_i^2 \geq cM^2$$

whenever $x_1, \ldots, x_{101}$ are real numbers such that $x_1 + \cdots + x_{101} = 0$ and M is the median of $x_1, \ldots, x_{101}$.

8 Find all real numbers k such that $r^4 + kr^3 + r^2 + 4kr + 16 = 0$ is true for exactly one real number r.

9 Given a, b, and c are complex numbers satisfying

$$a^2 + ab + b^2 = 1 + i$$
$$b^2 + bc + c^2 = -2$$
$$c^2 + ca + a^2 = 1,$$

compute $(ab + bc + ca)^2$. (Here, $i = \sqrt{-1}$)

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10 For an integer n, let $f_9(n)$ denote the number of positive integers $d \leq 9$ dividing n. Suppose that m is a positive integer and $b_1, b_2, \ldots, b_m$ are real numbers such that $f_9(n) = \sum_{j=1}^{m} b_j f_9(n-j)$ for all $n > m$. Find the smallest possible value of m.

– Combinatorics

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1 There are 100 students who want to sign up for the class Introduction to Acting. There are three class sections for Introduction to Acting, each of which will fit exactly 20 students. The 100 students, including Alex and Zhu, are put in a lottery, and 60 of them are randomly selected to fill up the classes. What is the probability that Alex and Zhu end up getting into the same section for the class?

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2 There are 10 people who want to choose a committee of 5 people among them. They do this by first electing a set of $1, 2, 3$, or 4 committee leaders, who then choose among the remaining people to complete the 5-person committee. In how many ways can the committee be formed, assuming that people are distinguishable? (Two committees that have the same members but different sets of leaders are considered to be distinct.)

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3 Bob writes a random string of 5 letters, where each letter is either A, B, C, or D. The letter in each position is independently chosen, and each of the letters A, B, C, D is chosen with equal probability. Given that there are at least two $A's$ in the string, find the probability that there are at least three $A's$ in the string.

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4 Find the number of triples of sets (A, B, C) such that:

(a) $A, B, C \subseteq \{1, 2, 3, \ldots, 8\}$.
(b) $|A \cap B| = |B \cap C| = |C \cap A| = 2$.
(c) $|A| = |B| = |C| = 4$.

Here, $|S|$ denotes the number of elements in the set S.

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5 Eli, Joy, Paul, and Sam want to form a company; the company will have 16 shares to split among the 4 people. The following constraints are imposed:

• Every person must get a positive integer number of shares, and all 16 shares must be given out. • No one person can have more shares than the other three people combined.

Assuming that shares are indistinguishable, but people are distinguishable, in how many ways can the shares be given out?

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6 We have a calculator with two buttons that displays and integer x. Pressing the first button replaces x by $\lfloor \frac{x}{2} \rfloor$, and pressing the second button replaces x by $4x+1$. Initially, the calculator displays 0. How many integers less than or equal to 2014 can be achieved through a sequence of arbitrary button presses? (It is permitted for the number displayed to exceed 2014 during the sequence. Here, $\lfloor y \rfloor$ denotes the greatest integer less than or equal to the real number y).

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7 Six distinguishable players are participating in a tennis tournament. Each player plays one match of tennis against every other player. The outcome of each tennis match is a win for one player and a loss for the other players; there are no ties. Suppose that whenever A and B are players in the tournament for which A won (strictly) more matches than B over the course of the tournament, it is also the case that A won the match against B during the tournament. In how many ways could the tournament have gone?

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8 The integers $1, 2, \ldots, 64$ are written in the squares of a 8×8 chess board, such that for each $1 \le i < 64$, the numbers i and $i+1$ are in squares that share an edge. What is the largest possible sum that can appear along one of the diagonals?

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9 There is a heads up coin on every integer of the number line. Lucky is initially standing on the zero point of the number line facing in the positive direction. Lucky performs the following procedure:

• Lucky looks at the coin (or lack thereof) underneath him.

• $-$ If the coin is heads, Lucky flips it to tails up, turns around, and steps forward a distance of one unit.

$a-$ If the coin is tails, Lucky picks up the coin and steps forward a distance of one unit facing the same direction.

$a-$ If there is no coin, Lucky places a coin heads up underneath him and steps forward a distance of one unit facing the same direction.

He repeats this procedure until there are 20 coins anywhere that are tails up. How many times has Lucky performed the procedure when the process stops?

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10 An *up-right path* from $(a, b) \in \mathbb{R}^2$ to $(c, d) \in \mathbb{R}^2$ is a finite sequence $(x_1, y_z), \ldots, (x_k, y_k)$ of points in $\mathbb{R}^2$ such that $(a, b) = (x_1, y_1), (c, d) = (x_k, y_k)$, and for each $1 \le i < k$ we have that either $(x_{i+1}, y_{y+1}) = (x_i + 1, y_i)$ or $(x_{i+1}, y_{i+1}) = (x_i, y_i + 1)$. Two up-right paths are said to intersect if they share any point.

Find the number of pairs (A, B) where A is an up-right path from $(0,0)$ to $(4,4)$, B is an up-right path from $(2,0)$ to $(6,4)$, and A and B do not intersect.

– Geometry

1 Let O_1 and O_2 be concentric circles with radii 4 and 6, respectively. A chord AB is drawn in O_1 with length 2. Extend AB to intersect O_2 in points C and D. Find CD.

2 Point P and line ℓ are such that the distance from P to ℓ is 12. Given that T is a point on ℓ such that $PT = 13$, find the radius of the circle passing through P and tangent to ℓ at T.

3 ABC is a triangle such that $BC = 10$, $CA = 12$. Let M be the midpoint of side AC. Given that BM is parallel to the external bisector of $\angle A$, find area of triangle ABC. (Lines AB and AC form two angles, one of which is $\angle BAC$. The external angle bisector of $\angle A$ is the line that bisects the other angle.

4 In quadrilateral $ABCD$, $\angle DAC = 98^\circ$, $\angle DBC = 82^\circ$, $\angle BCD = 70^\circ$, and $BC = AD$. Find $\angle ACD$.

5 Let $\mathcal{C}$ be a circle in the xy plane with radius 1 and center $(0,0,0)$, and let P be a point in space with coordinates $(3,4,8)$. Find the largest possible radius of a sphere that is contained entirely in the slanted cone with base $\mathcal{C}$ and vertex P.

6 In quadrilateral $ABCD$, we have $AB = 5$, $BC = 6$, $CD = 5$, $DA = 4$, and $\angle ABC = 90^\circ$. Let AC and BD meet at E. Compute $\dfrac{BE}{ED}$.

7 Triangle ABC has sides $AB = 14$, $BC = 13$, and $CA = 15$. It is inscribed in circle Γ, which has center O. Let M be the midpoint of AB, let B' be the point on Γ diametrically opposite B, and let X be the intersection of AO and MB'. Find the length of AX.

8 Let ABC be a triangle with sides $AB = 6$, $BC = 10$, and $CA = 8$. Let M and N be the midpoints of BA and BC, respectively. Choose the point Y on ray CM so that the circumcircle of triangle AMY is tangent to AN. Find the area of triangle NAY.

9 Two circles are said to be *orthogonal* if they intersect in two points, and their tangents at either point of intersection are perpendicular. Two circles ω_1 and ω_2 with radii 10 and 13, respectively, are externally tangent at point P. Another circle ω_3 with radius $2\sqrt{2}$ passes through P and is orthogonal to both ω_1 and ω_2. A fourth circle ω_4, orthogonal to ω_3, is externally tangent to ω_1 and ω_2. Compute the radius of ω_4.

10 Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let Γ be the circumcircle of ABC, let O be its circumcenter, and let M be the midpoint of minor arc BC. Circle ω_1 is internally

tangent to Γ at A, and circle ω_2, centered at M, is externally tangent to ω_1 at a point T. Ray AT meets segment BC at point S, such that $BS - CS = \dfrac{4}{15}$. Find the radius of ω_2

– Guts

1 [4] Compute the prime factorisation of 159999.

2 [4] Let $x_1, x_2, \ldots, x_{100}$ be defined so that for each i, x_i is a (uniformly) random integer between 1 and 6 inclusive. Find the expected number of integers in the set $\{x_1, x_1 + x_2, \ldots, x_1 + x_2 + \cdots + x_{100}\}$ that are multiples of 6.

3 [4] Let $ABCDEF$ be a regular hexagon. Let P be the circle inscribed in $\triangle BDF$. Find the ratio of the area of circle P to the area of rectangle $ABDE$.

4 [4] Let D be the set of divisors of 100. Let Z be the set of integers between 1 and 100, inclusive. Mark chooses an element d of D and an element z of Z uniformly at random. What is the probability that d divides z?

5 [5] If four fair six-sided dice are rolled, what is the probability that the lowest number appearing on any die is exactly 3?

6 [5] Find all integers n for which $\frac{n^3+8}{n^2-4}$ is an integer.

7 The Evil League of Evil is plotting to poison the city's water supply. They plan to set out from their headquarters at $(5, 1)$ and put poison in two pipes, one along the line $y = x$ and one along the line $x = 7$. However, they need to get the job done quickly before Captain Hammer catches them. What's the shortest distance they can travel to visit both pipes and then return to their headquarters?

8 The numbers $2^0, 2^1, \ldots, 2^{15}, 2^{16} = 65536$ are written on a blackboard. You repeatedly take two numbers on the blackboard, subtract one form the other, erase them both, and write the result of the subtraction on the blackboard. What is the largest possible number that can remain on the blackboard when there is only one number left?

9 Compute the side length of the largest cube contained in the region

$$\{(x, y, z) : x^2 + y^2 + z^2 \le 25 \text{ and } x, y \ge 0\}$$

of three-dimensional space.

10 [6] Find the number of sets $\mathcal{F}$ of subsets of the set $\{1, \ldots, 2014\}$ such that:

a) For any subsets $S_1, S_2 \in \mathcal{F}$, $S_1 \cap S_2 \in \mathcal{F}$.
b) If $S \in \mathcal{F}$, $T \subseteq \{1, \ldots, 2014\}$, and $S \subseteq T$, then $T \in \mathcal{F}$.

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11 Two fair octahedral dice, each with the numbers 1 through 8 on their faces, are rolled. Let N be the remainder when the product of the numbers showing on the two dice is divided by 8. Find the expected value of N.

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12 Find a nonzero monic polynomial $P(x)$ with integer coefficients and minimal degree such that $P(1 - \sqrt[3]{2} + \sqrt[3]{4}) = 0$. (A polynomial is called *monic* if its leading coefficient is 1.)

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13 An auditorium has two rows of seats, with 50 seats in each row. 100 indistinguishable people sit in the seats one at a time, subject to the condition that each person, except for the first person to sit in each row, must sit to the left or right of an occupied seat, and no two people can sit in the same seat. In how many ways can this process occur?

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14 Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $\angle D = 90^\circ$. Suppose that there is a point E on CD such that $AE = BE$ and that triangles AED and CEB are similar, but not congruent. Given that $\frac{CD}{AB} = 2014$, find $\frac{BC}{AD}$.

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15 Given a regular pentagon of area 1, a pivot line is a line not passing through any of the pentagon's vertices such that there are 3 vertices of the pentagon on one side of the line and 2 on the other. A pivot point is a point inside the pentagon with only finitely many non-pivot lines passing through it. Find the area of the region of pivot points.

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16 Suppose that x and y are positive real numbers such that $x^2 - xy + 2y^2 = 8$. Find the maximum possible value of $x^2 + xy + 2y^2$.

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17 Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following conditions:

(a) $f(1) = 1$.
(b) $f(a) \leq f(b)$ whenever a and b are positive integers with $a \leq b$.
(c) $f(2a) = f(a) + 1$ for all positive integers a.

How many possible values can the 2014-tuple $(f(1), f(2), \ldots, f(2014))$ take?

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18 Find the number of ordered quadruples of positive integers (a, b, c, d) such that a, b, c, and d are all (not necessarily distinct) factors of 30 and $abcd > 900$.

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19 Let $ABCD$ be a trapezoid with $AB \parallel CD$. The bisectors of $\angle CDA$ and $\angle DAB$ meet at E, the bisectors of $\angle ABC$ and $\angle BCD$ meet at F, the bisectors of $\angle BCD$ and $\angle CDA$ meet at G, and the bisectors of $\angle DAB$ and $\angle ABC$ meet at H. Quadrilaterals $EABF$ and $EDCF$ have areas 24 and 36, respectively, and triangle ABH has area 25. Find the area of triangle CDG.

20 A deck of 8056 cards has 2014 ranks numbered 12014. Each rank has four suits - hearts, diamonds, clubs, and spades. Each card has a rank and a suit, and no two cards have the same rank and the same suit. How many subsets of the set of cards in this deck have cards from an odd number of distinct ranks?

21 Compute the number of ordered quintuples of nonnegative integers $(a_1, a_2, a_3, a_4, a_5)$ such that $0 \leq a_1, a_2, a_3, a_4, a_5 \leq 7$ and 5 divides $2^{a_1} + 2^{a_2} + 2^{a_3} + 2^{a_4} + 2^{a_5}$.

22 Let ω be a circle, and let $ABCD$ be a quadrilateral inscribed in ω. Suppose that BD and AC intersect at a point E. The tangent to ω at B meets line AC at a point F, so that C lies between E and F. Given that $AE = 6$, $EC = 4$, $BE = 2$, and $BF = 12$, find DA.

23 Let $S = \{-100, -99, -98, \ldots, 99, 100\}$. Choose a 50-element subset T of S at random. Find the expected number of elements of the set $\{|x| : x \in T\}$.

24 Let $A = \{a_1, a_2, \ldots, a_7\}$ be a set of distinct positive integers such that the mean of the elements of any nonempty subset of A is an integer. Find the smallest possible value of the sum of the elements in A.

25 Let ABC be an equilateral triangle of side length 6 inscribed in a circle ω. Let A_1, A_2 be the points (distinct from A) where the lines through A passing through the two trisection points of BC meet ω. Define B_1, B_2, C_1, C_2 similarly. Given that $A_1, A_2, B_1, B_2, C_1, C_2$ appear on ω in that order, find the area of hexagon $A_1A_2B_1B_2C_1C_2$.

26 For $1 \leq j \leq 2014$, define

$$b_j = j^{2014} \prod_{i=1, i \neq j}^{2014} (i^{2014} - j^{2014})$$

where the product is over all $i \in \{1, \ldots, 2014\}$ except $i = j$. Evaluate

$$\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_{2014}}.$$

27 Suppose that $(a_1, \ldots, a_{20})$ and $(b_1, \ldots, b_{20})$ are two sequences of integers such that the sequence $(a_1, \ldots, a_{20}, b_1, \ldots, b_{20})$ contains each of the numbers $1, \ldots, 40$ exactly once. What is the maximum possible value of the sum

$$\sum_{i=1}^{20} \sum_{j=1}^{20} \min(a_i, b_j)?$$

28 Let $f(n)$ and $g(n)$ be polynomials of degree 2014 such that $f(n) + (-1)^n g(n) = 2^n$ for $n = 1, 2, \ldots, 4030$. Find the coefficient of x^{2014} in $g(x)$.

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29 Natalie has a copy of the unit interval $[0, 1]$ that is colored white. She also has a black marker, and she colors the interval in the following manner: at each step, she selects a value $x \in [0, 1]$ uniformly at random, and

(a) If $x \leq \frac{1}{2}$ she colors the interval $[x, x + \frac{1}{2}]$ with her marker.

(b) If $x > \frac{1}{2}$ she colors the intervals $[x, 1]$ and $[0, x - \frac{1}{2}]$ with her marker.

What is the expected value of the number of steps Natalie will need to color the entire interval black?

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30 Let ABC be a triangle with circumcenter O, incenter I, $\angle B = 45^\circ$, and $OI \parallel BC$. Find $\cos \angle C$.

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31 Compute

$$\sum_{k=1}^{1007} \left(\cos \left(\frac{\pi k}{1007} \right) \right)^{2014}.$$

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32 Find all ordered pairs (a, b) of complex numbers with $a^2 + b^2 \neq 0$, $a + \frac{10b}{a^2+b^2} = 5$, and $b + \frac{10a}{a^2+b^2} = 4$.

– Team

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1 Let ω be a circle, and let A and B be two points in its interior. Prove that there exists a circle passing through A and B that is contained in the interior of ω.

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2 Let $a_1, a_2, \ldots$ be an infinite sequence of integers such that a_i divides a_{i+1} for all $i \geq 1$, and let b_i be the remainder when a_i is divided by 210. What is the maximal number of distinct terms in the sequence $b_1, b_2, \ldots$?

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3 There are n girls $G_1, \ldots, G_n$ and n boys $B_1, \ldots, B_n$. A pair (G_i, B_j) is called *suitable* if and only if girl G_i is willing to marry boy B_j. Given that there is exactly one way to pair each girl with a distinct boy that she is willing to marry, what is the maximal possible number of suitable pairs?

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4 Compute

$$\sum_{k=0}^{100} \left\lfloor \frac{2^{100}}{2^{50} + 2^k} \right\rfloor.$$

(Here, if x is a real number, then $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

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5 Prove that there exists a nonzero complex number c and a real number d such that

$$\left|\left|\frac{1}{1+z+z^2}\right|-\left|\frac{1}{1+z+z^2}-c\right|\right|=d$$

for all z with $|z|=1$ and $1+z+z^2\neq 0$. (Here, $|z|$ denotes the absolute value of the complex number z, so that $|a+bi|=\sqrt{a^2+b^2}$ for real numbers a,b.)

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6 Let n be a positive integer. A sequence $(a_0,\ldots,a_n)$ of integers is *acceptable* if it satisfies the following conditions:

- $0=|a_0|<|a_1|<\cdots<|a_{n-1}|<|a_n|$.
-The sets $\{|a_1-a_0|,|a_2-a_1|,\ldots,|a_{n-1}-a_{n-2}|,|a_n-a_{n-1}|\}$ and $\{1,3,9,\ldots,3^{n-1}\}$ are equal.

Prove that the number of acceptable sequences of integers is $(n+1)!$.

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7 Find the maximum possible number of diagonals of equal length in a convex hexagon.

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8 Let ABC be an acute triangle with circumcenter O such that $AB=4$, $AC=5$, and $BC=6$. Let D be the foot of the altitude from A to BC, and E be the intersection of AO with BC. Suppose that X is on BC between D and E such that there is a point Y on AD satisfying $XY\parallel AO$ and $YO\perp AX$. Determine the length of BX.

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9 For integers $m,n\geq 1$, let $A(n,m)$ be the number of sequences $(a_1,\cdots,a_{nm})$ of integers satisfying the following two properties:

-Each integer k with $1\leq k\leq n$ occurs exactly m times in the sequence $(a_1,\cdots,a_{nm})$.
-If i,j, and k are integers such that $1\leq i\leq nm$ and $1\leq j\leq k\leq n$, then j occurs in the sequence $(a_1,\cdots,a_i)$ at least as many times as k does.

For example, if $n=2$ and $m=5$, a possible sequence is $(a_1,\cdots,a_{10})=(1,1,2,1,2,2,1,2,1,2)$. On the other hand, the sequence $(a_1,\cdots,a_{10})=(1,2,1,2,2,1,1,1,2,2)$ does not satisfy property (2) for $i=5$, $j=1$, and $k=2$.

Prove that $A(n,m)=A(m,n)$.

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10 Fix a positive real number $c>1$ and positive integer n. Initially, a blackboard contains the numbers $1,c,\ldots,c^{n-1}$. Every minute, Bob chooses two numbers a,b on the board and replaces them with $ca+c^2b$. Prove that after $n-1$ minutes, the blackboard contains a single number no less than

$$\left(\frac{c^{n/L}-1}{c^{1/L}-1}\right)^L,$$

where $\phi=\frac{1+\sqrt{5}}{2}$ and $L=1+\log_\phi(c)$.