Serbia National Math Olympiad 2007

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Day 1

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1 A point D is chosen on the side AC of a triangle ABC with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incenter of triangle BCD. Prove that the line KL intersects the line segment AJ at its midpoint.

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2 Triangle ΔGRB is dissected into 25 small triangles as shown. All vertices of these triangles are painted in three colors so that the following conditions are satisfied: Vertex G is painted in green, vertex R in red, and B in blue; Each vertex on side GR is either green or red, each vertex on RB is either red or blue, and each vertex on GB is either green or blue. The vertices inside the big triangle are arbitrarily colored.

Prove that, regardless of the way of coloring, at least one of the 25 small triangles has vertices of three different colors.

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3 Determine all pairs of natural numbers $(x; n)$ that satisfy the equation

$$x^3 + 2x + 1 = 2^n.$$

Day 2

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1 Let k be a natural number. For each function $f : \mathbb{N} \to \mathbb{N}$ define the sequence of functions $(f_m)_{m\geq 1}$ by $f_1 = f$ and $f_{m+1} = f \circ f_m$ for $m \geq 1$. Function f is called k-*nice* if for each $n \in \mathbb{N} : f_k(n) = f(n)^k$.

(a) For which k does there exist an injective k-nice function f ?

(b) For which k does there exist a surjective k-nice function f ?

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2 In a scalene triangle ABC, AD, BE, CF are the angle bisectors $(D \in BC, E \in AC, F \in AB)$. Points K_a, K_b, K_c on the incircle of triangle ABC are such that DK_a, EK_b, FK_c are tangent to the incircle and $K_a \notin BC, K_b \notin AC, K_c \notin AB$. Let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB , respectively. Prove that the lines A_1K_a, B_1K_b, C_1K_c intersect on the incircle of triangle ABC.

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3 Let k be a given natural number. Prove that for any positive numbers $x; y; z$ with the sum 1 the following inequality holds:

$$\frac{x^{k+2}}{x^{k+1}+y^k+z^k}+\frac{y^{k+2}}{y^{k+1}+z^k+x^k}+\frac{z^{k+2}}{z^{k+1}+x^k+y^k}\geq\frac{1}{7}.$$

When does equality occur?