Serbia National Math Olympiad 2008

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Day 1

1	Find all nonegative integers x, y, z such that $12^x + y^4 = 2008^z$
2	Triangle $\triangle ABC$ is given. Points D i E are on line AB such that $D - A - B - E, AD = AC$ and $BE = BC$. Bisector of internal angles at A and B intersect BC, AC at P and Q, and circumcircle of ABC at M and N. Line which connects A with center of circumcircle of BME and line which connects B and center of circumcircle of AND intersect at X. Prove that $CX \perp PQ$.
3	Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove inequality: $$\frac{1}{bc + a + \frac{1}{a}} + \frac{1}{ac + b + \frac{1}{b}} + \frac{1}{ab + c + \frac{1}{c}} \leqslant \frac{27}{31}.$$

Day 2

4	Each point of a plane is painted in one of three colors. Show that there exists a triangle such that: (i) all three vertices of the triangle are of the same color; (ii) the radius of the circumcircle of the triangle is 2008; (iii) one angle of the triangle is either two or three times greater than one of the other two angles.
5	The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 3$, $a_2 = 11$ and $a_n = 4a_{n-1} - a_{n-2}$, for $n \geq 3$. Prove that each term of this sequence is of the form $a^2 + 2b^2$ for some natural numbers a and b.
6	In a convex pentagon $ABCDE$, let $\angle EAB = \angle ABC = 120^\circ$, $\angle ADB = 30^\circ$ and $\angle CDE = 60^\circ$. Let $AB = 1$. Prove that the area of the pentagon is less than $\sqrt{3}$.