Serbia National Math Olympiad 2011

www.artofproblemsolving.com/community/c3636

by RaleD

Day 1

1 Let $n \geq 2$ be integer. Let a_0, a_1, ... a_n be sequence of positive reals such that: $(a_{k-1}+a_k)(a_k+a_{k+1})=a_{k-1}-a_{k+1}$, for $k=1,2,...,n-1$.
Prove $a_n < \frac{1}{n-1}$.

2 Let n be an odd positive integer such that both $\phi(n)$ and $\phi(n+1)$ are powers of two. Prove $n+1$ is power of two or $n=5$.

3 Let H be orthocenter and O circumcenter of an acuted angled triangle ABC. D and E are feets of perpendiculars from A and B on BC and AC respectively. Let OD and OE intersect BE and AD in K and L, respectively. Let X be intersection of circumcircles of HKD and HLE different than H, and M is midpoint of AB. Prove that K, L, M are collinear iff X is circumcenter of EOD.

Day 2

1 On sides AB, AC, BC are points M, X, Y, respectively, such that $AX = MX$; $BY = MY$. K, L are midpoints of AY and BX. O is circumcenter of ABC, O_1, O_2 are symmetric with O with respect to K and L. Prove that X, Y, O_1, O_2 are concyclic.

2 Are there positive integers a, b, c greater than 2011 such that: $(a+\sqrt{b})^c = ...2010,2011...$?

3 Set T consists of 66 points in plane, and P consists of 16 lines in plane. Pair (A,l) is *good* if $A \in T$, $l \in P$ and $A \in l$. Prove that maximum number of good pairs is no greater than 159, and prove that there exits configuration with exactly 159 good pairs.