Serbia National Math Olympiad 2012

www.artofproblemsolving.com/community/c3637

by Djile

Day 1

---

1 Let $ABCD$ be a parallelogram and P be a point on diagonal BD such that $\angle PCB = \angle ACD$. Circumcircle of triangle ABD intersects line AC at points A and E. Prove that

$$\angle AED = \angle PEB.$$

---

2 Find all natural numbers a and b such that

$$a|b^2, \quad b|a^2 \text{ and } a+1|b^2+1.$$

---

3 A fly and k spiders are placed in some vertices of 2012×2012 lattice. One move consists of following: firstly, fly goes to some adjacent vertex or stays where it is and then every spider goes to some adjacent vertex or stays where it is (more than one spider can be in the same vertex). Spiders and fly knows where are the others all the time.
a) Find the smallest k so that spiders can catch the fly in finite number of moves, regardless of their initial position.
b) Answer the same question for three-dimensional lattice $2012 \times 2012 \times 2012$.

(Vertices in lattice are adjacent if exactly one coordinate of one vertex is different from the same coordinate of the other vertex, and their difference is equal to 1. Spider catches a fly if they are in the same vertex.)

Day 2

---

1 Find all natural numbers n for which there is a permutation $(p_1, p_2, ..., p_n)$ of numbers $(1, 2, ..., n)$ such that sets $\{p_1+1, p_2+2, ..., p_n+n\}$ and $\{p_1-1, p_2-2, ..., p_n-n\}$ are complete residue systems $\mod n$.

---

2 Let $\mathbb{K}$ be two-dimensional integer lattice. Is there a bijection $f : \mathbb{N} \to \mathbb{K}$, such that for every distinct $a, b, c \in \mathbb{N}$ we have:

$$\gcd(a,b,c) > 1 \Rightarrow f(a), f(b), f(c) \text{ are not colinear?}$$

---

3 We are given $n > 1$ piles of coins. There are two different types of coins: real and fake coins; they all look alike, but coins of the same type have the same mass, while the coins from different types have different masses. Coins that belong to the same pile are of the same type. We know the mass of real coin.

Find the minimal number of weightings on digital scale that we need in order to conclude: which piles consists of which type of coins and also the mass of fake coin.

(We assume that every pile consists from infinite number of coins.)