Serbia National Math Olympiad 2013

www.artofproblemsolving.com/community/c3638

by Djile

1 Let k be a natural number. Bijection $f:\mathbb{Z}\to\mathbb{Z}$ has the following property: for any integers i and j, $|i-j|\le k$ implies $|f(i)-f(j)|\le k$. Prove that for every $i,j\in\mathbb{Z}$ it stands:

$$|f(i)-f(j)|=|i-j|.$$

---

2 For a natural number n, set S_n is defined as:

$$S_n=\left\{\binom{n}{n},\binom{2n}{n},\binom{3n}{n},...,\binom{n^2}{n}\right\}.$$

a) Prove that there are infinitely many composite numbers n, such that the set S_n is not complete residue system mod n;

b) Prove that there are infinitely many composite numbers n, such that the set S_n is complete residue system mod n.

---

3 Let M, N and P be midpoints of sides BC, AC and AB, respectively, and let O be circumcenter of acute-angled triangle ABC. Circumcircles of triangles BOC and MNP intersect at two different points X and Y inside of triangle ABC. Prove that

$$\angle BAX=\angle CAY.$$

---

4 Determine all natural numbers n for which there is a partition of $\{1,2,...,3n\}$ in n pairwise disjoint subsets of the form $\{a,b,c\}$, such that numbers $b-a$ and $c-b$ are different numbers from the set $\{n-1,n,n+1\}$.

---

5 Let A' and B' be feet of altitudes from A and B, respectively, in acute-angled triangle ABC ($AC\neq BC$). Circle k contains points A' and B' and touches segment AB in D. If triangles ADA' and BDB' have the same area, prove that

$$\angle A'DB'=\angle ACB.$$

---

6 Find the largest constant $K \in \mathbb{R}$ with the following property:
if $a_1, a_2, a_3, a_4 > 0$ are numbers satisfying $a_i^2 + a_j^2 + a_k^2 \geq 2(a_ia_j + a_ja_k + a_ka_i)$, for every $1 \leq i < j < k \leq 4$, then

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 \geq K(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4).$$