Serbia Team Selection Test 2009

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by Bugi

Day 1

1   Let α and β be the angles of a non-isosceles triangle ABC at points A and B, respectively. Let the bisectors of these angles intersect opposing sides of the triangle in D and E, respectively. Prove that the acute angle between the lines DE and AB isn't greater than $\frac{|\alpha-\beta|}{3}$.

2   Find the least number which is divisible by 2009 and its sum of digits is 2009.

3   Find the largest natural number n for which there exist different sets $S_1, S_2, \ldots, S_n$ such that: $1°\ |S_i \cup S_j| \le 2004$ for each two $1 \le i, j \le n$ and $2°\ S_i \cup S_j \cup S_k = \{1, 2, \ldots, 2008\}$ for each three integers $1 \le i < j < k \le n$.

Day 2

1   Let $n \in \mathbb{N}$ and A_n set of all permutations $(a_1, \ldots, a_n)$ of the set $\{1, 2, \ldots, n\}$ for which

$$k | 2(a_1 + \cdots + a_k), \text{ for all } 1 \le k \le n.$$

Find the number of elements of the set A_n.

Proposed by Vidan Govedarica, Serbia

2   Let x, y, z be positive real numbers such that $xy + yz + zx = x + y + z$. Prove the inequality
$\frac{1}{x^2+y+1} + \frac{1}{y^2+z+1} + \frac{1}{z^2+x+1} \le 1$

When does the equality hold?

3   Let k be the inscribed circle of non-isosceles triangle $\triangle ABC$, which center is S. Circle k touches sides BC, CA, AB in points P, Q, R respectively. Line QR intersects BC in point M. Let a circle which contains points B and C touch k in point N. Circumscribed circle of $\triangle MNP$ intersects line AP in point L, different from P. Prove that points S, L and M are collinear.