Regional Olympiad - Federation Of Bosnia And Herzegovina 2008

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– First Grades

1 Squares BCA_1A_2 , CAB_1B_2 , ABC_1C_2 are outwardly drawn on sides of triangle $\triangle ABC$. If $AB_1A'C_2$, $BC_1B'A_2$, $CA_1C'B_2$ are parallelograms then prove that:

(i) Lines BC and AA' are orthogonal.

(ii)Triangles $\triangle ABC$ and $\triangle A'B'C'$ have common centroid

2 For arbitrary reals x, y and z prove the following inequality:
$x^2 + y^2 + z^2 - xy - yz - zx \geq \max\{\frac{3(x-y)^2}{4}, \frac{3(y-z)^2}{4}, \frac{3(y-z)^2}{4}\}$

3 Let b be an even positive integer. Assume that there exist integer $n > 1$ such that $\frac{b^n-1}{b-1}$ is perfect square.
Prove that b is divisible by 8.

4 Given are two disjoint sets A and B such that their union is $\mathbb{N}$. Prove that for all positive integers n there exist different numbers a and b, both greater than n, such that either $\{a, b, a + b\}$ is contained in A or $\{a, b, a + b\}$ is contained in B.

– Second Grades

1 Given is an acute angled triangle $\triangle ABC$ with side lengths a, b and c (in an usual way) and circumcenter O. Angle bisector of angle $\angle BAC$ intersects circumcircle at points A and A_1. Let D be projection of point A_1 onto line AB, L and M be midpoints of AC and AB , respectively.

(i) Prove that $AD = \frac{1}{2}(b + c)$

(ii) If triangle $\triangle ABC$ is an acute angled prove that $A_1D = OM + OL$

2 IF a, b and c are positive reals such that $a^2 + b^2 + c^2 = 1$ prove the inequality:

$$\frac{a^5 + b^5}{ab(a + b)} + \frac{b^5 + c^5}{bc(b + c)} + \frac{c^5 + a^5}{ca(a + b)} \geq 3(ab + bc + ca) - 2.$$

3 Prove that equation $p^4+q^4=r^4$ does not have solution in set of prime numbers.

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4 n points (no three being collinear) are given in a plane. Some points are connected and they form k segments. If no three of these segments form triangle (equiv. there are no three points, such that each two of them are connected) prove that $k \leq \left\lfloor \frac{n^2}{4} \right\rfloor$

– Third Grades

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1 Two circles with centers S_1 and S_2 are externally tangent at point K. These circles are also internally tangent to circle S at points A_1 and A_2, respectively. Denote by Pone of the intersection points of S and common tangent to S_1 and S_2 at K.Line PA_1 intersects S_1 at B_1 while PA_2 intersects S_2 at B_2.
Prove that B_1B_2 is common tangent of circles S_1 and S_2.

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2 If a, b and c are positive reals prove inequality:

$$\left(1+\frac{4a}{b+c}\right)\left(1+\frac{4b}{a+c}\right)\left(1+\frac{4c}{a+b}\right)>25.$$

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3 Find all positive integers a and b such that $\frac{a^4+a^3+1}{a^2b^2+ab^2+1}$ is an integer.

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4 A rectangular table 9 rows $\times$ 2008 columns is fulfilled with numbers 1, 2, ...,2008 in a such way that each number appears exactly 9 times in table and difference between any two numbers from same column is not greater than 3. What is maximum value of minimum sum in column (with minimal sum)?

– Fourth Grades

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1 Given are three pairwise externally tangent circles K_1 , K_2 and K_3. denote by P_1 tangent point of K_2 and K_3 and by P_2 tangent point of K_1 and K_3.

Let AB (A and B are different from tangency points) be a diameter of circle K_3. Line AP_2 intersects circle K_1 (for second time) at point X and line BP_1 intersects circle K_2(for second time) at Y.

If Z is intersection point of lines AP_1 and BP_2 prove that points X, Y and Z are collinear.

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2 Find all positive integers a and b such that $\frac{a^4+a^3+1}{a^2b^2+ab^2+1}$ is an integer.

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3 A rectangular table 9 rows $\times$ 2008 columns is fulfilled with numbers 1, 2, ...,2008 in a such way that each number appears exactly 9 times in table and difference between any two numbers from same column is not greater than 3. What is maximum value of minimum sum in column (with minimal sum)?

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4 Determine is there a function $a : \mathbb{N} \rightarrow \mathbb{N}$ such that: $i)$ $a(0) = 0$ $ii)$ $a(n) = n - a(a(n))$, $\forall n \in \mathbb{N}$. If exists prove: $a)$ $a(k) \geq a(k-1)$ $b)$ Does not exist positive integer k such that $a(k-1) = a(k) = a(k+1)$.
