Mongolia Team Selection Test 2008

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Day 1

1 Given acute angle triangle ABC. Let CDbe the altitude , H be the orthocenter and O be the circumcenter of $\triangle ABC$ The line through point D and perpendicular with OD , is intersect BC at E. Prove that $\angle DHE = \angle ABC$.

2 Let a,b,c,d be the positive integers such that $a > b > c > d$ and $(a+b-c+d)|(ac+bd)$. Prove that if m is arbitrary positive integer , n is arbitrary odd positive integer, then $a^n b^m + c^m d^n$ is composite number

3 Given a circumscribed trapezium $ABCD$ with circumcircle ω and 2 parallel sides AD, BC ($BC < AD$). Tangent line of circle ω at the point C meets with the line AD at point P. PE is another tangent line of circle ω and $E \in \omega$. The line BP meets circle ω at point K. The line passing through the point C paralel to AB intersects with AE and AK at points N and M respectively. Prove that M is midpoint of segment CN.

Day 2

1 Find all function $f: R^+ \rightarrow R^+$ such that for any $x,y,z \in R^+$ such that $x+y \geq z$, $f(x+y-z) + f(2\sqrt{xz}) + f(2\sqrt{yz}) = f(x+y+z)$

2 Given positive integersm,n such that $m < n$. Integers $1, 2, ..., n^2$ are arranged in $n \times n$ board. In each row, m largest number colored red. In each column m largest number colored blue. Find the minimum number of cells such that colored both red and blue.

3 Find the maximum number C such that for any nonnegative x,y,z the inequality $x^3+y^3+z^3+C(xy^2+yz^2+zx^2) \geq (C+1)(x^2y+y^2z+z^2x)$ holds.

Day 3

1 Given an integer a. Let p is prime number such that $p|a$ and $p \equiv \pm 3 (mod 8)$. Define a sequence $\{a_n\}_{n=0}^{\infty}$ such that $a_n = 2^n + a$. Prove that the sequence $\{a_n\}_{n=0}^{\infty}$ has finitely number of square of integer.

2 The quadrilateral $ABCD$ inscribed in a circle wich has diameter BD. Let A', B' are symmetric to A, B with respect to the line BD and AC respectively. If $A'C \cap BD = P$ and $AC \cap B'D = Q$ then prove that $PQ \perp AC$

3 Given positive integers $m, n > 1$. Prove that the equation $(x+1)^n + (x+2)^n + ... + (x+m)^n = (y+1)^{2n} + (y+2)^{2n} + ... + (y+m)^{2n}$ has finitely number of solutions $x, y \in N$

Day 4

1 How many ways to fill the board 4×4 by nonnegative integers, such that sum of the numbers of each row and each column is 3?

2 Let $a_1, a_2, ..., a_n$ is permutaion of $1, 2, ..., n$. For this permutaion call the pair (a_i, a_j) *wrong pair* if $i < j$ and $a_i > a_j$.Let *number of inversion* is number of *wrong pair* of permutation $a_1, a_2, a_3, .., a_n$. Let $n \geq 2$ is positive integer. Find the number of permutation of $1, 2, .., n$ such that its *number of inversion* is divisible by n.

3 Let Ω is circle with radius R and center O. Let ω is a circle inside of the Ω with center I radius r. X is variable point of ω and tangent line of ω pass through X intersect the circle Ω at points A, B. A line pass through X perpendicular with AI intersect ω at Y distinct with X.Let point C is symmetric to the point I with respect to the line XY.Find the locus of circumcenter of triangle ABC when X varies on ω