Mongolia Team Selection Test 2011

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by Bacteria

Day 1

1 Let $v(n)$ be the order of 2 in $n!$. Prove that for any positive integers a and m there exists n ($n>1$) such that $v(n) \equiv a(\mod m)$.

I have a book with Mongolian problems from this year, and this problem appeared in it. Perhaps I am terribly misinterpreting this problem, but it seems like it is wrong. Any ideas?

2 Mongolia TST 2011 Test 1#2

Let p be a prime number. Prove that:

$\sum_{k=0}^{p}(-1)^k\binom{p}{k}\binom{p+k}{k} \equiv -1(\mod p^3)$

(proposed by B. Batbayasgalan, inspired by Putnam olympiad problem)

Note: I believe they meant to say $p>2$ as well.

3 We are given an acute triangle ABC. Let (w,I) be the inscribed circle of ABC, (Ω,O) be the circumscribed circle of ABC, and A_0 be the midpoint of altitude AH. w touches BC at point D. A_0D and w intersect at point P, and the perpendicular from I to A_0D intersects BC at the point M. MR and MS lines touch Ω at R and S respectively [note: I am not entirely sure of what is meant by this, but I am pretty sure it means draw the tangents to Ω from M]. Prove that the points R,P,D,S are concyclic.
(proposed by E. Enkzaya, inspired by Vietnamese olympiad problem)

Day 2

1 A group of the pupils in a class are called *dominant* if any other pupil from the class has a friend in the group. If it is known that there exist at least 100 dominant groups, prove that there exists at least one more dominant group.

(proposed by B. Batbayasgalan, inspired by Komal problem)

2 Let ABC be a scalene triangle. The inscribed circle of ABC touches the sides BC, CA, and AB at the points A_1, B_1, C_1 respectively. Let I be the incenter, O be the circumcenter, and lines OI and BC meet at point D. The perpendicular line from A_1 to B_1C_1 intersects AD at point E. Prove that B_1C_1 passes through the midpoint of EA_1.

3 Let G be a graph, not containing K_4 as a subgraph and $|V(G)| = 3k$ (I interpret this to be the number of vertices is divisible by 3). What is the maximum number of triangles in G?

Day 3

1 Let $A = \{a^2 + 13b^2 \mid a, b \in \mathbb{Z}, b \neq 0\}$. Prove that there
a) exist
b) exist infinitely many x, y integer pairs such that $x^{13} + y^{13} \in A$ and $x + y \notin A$.

(proposed by B. Bayarjargal)

2 Given a triangle ABC, the internal and external bisectors of angle A intersect BC at points D and E respectively. Let F be the point (different from A) where line AC intersects the circle w with diameter DE. Finally, draw the tangent at A to the circumcircle of triangle ABF, and let it hit w at A and G. Prove that $AF = AG$.

3 Let n and d be positive integers satisfying $d < \frac{n}{2}$. There are n boys and n girls in a school. Each boy has at most d girlfriends and each girl has at most d boyfriends. Prove that one can introduce some of them to make each boy have exactly $2d$ girlfriends and each girl have exactly $2d$ boyfriends. (I think we assume if a girl has a boyfriend, she is his girlfriend as well and vice versa)

(proposed by B. Batbaysgalan, folklore).

Day 4

1 Let t, k, m be positive integers and $t > \sqrt{km}$. Prove that $\binom{2m}{0} + \binom{2m}{1} + \cdots + \binom{2m}{m-t-1} < \frac{2^{2m}}{2k}$

(proposed by B. Amarsanaa, folklore)

2 Let r be a given positive integer. Is is true that for every r-colouring of the natural numbers there exists a monochromatic solution of the equation $x + y = 3z$?

(proposed by B. Batbaysgalan, folklore)

3 Let m and n be positive integers such that $m > n$ and $m \equiv n \pmod 2$. If $(m^2 - n^2 + 1) \mid n^2 - 1$, then prove that $m^2 - n^2 + 1$ is a perfect square.

(proposed by G. Batzaya, folklore)