National Science Olympiad 2005

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by jgnr

Day 1

1 Let n be a positive integer. Determine the number of triangles (non congruent) with integral side lengths and the longest side length is n.

2 For an arbitrary positive integer n, define $p(n)$ as the product of the digits of n (in decimal). Find all positive integers n such that $11p(n) = n^2 - 2005$.

3 Let k and m be positive integers such that $\frac{1}{2}\left(\sqrt{k+4\sqrt{m}}-\sqrt{k}\right)$ is an integer.

(a) Prove that $\sqrt{k}$ is rational.

(b) Prove that $\sqrt{k}$ is a positive integer.

4 Let M be a point in triangle ABC such that $\angle AMC = 90^\circ$, $\angle AMB = 150^\circ$, $\angle BMC = 120^\circ$. The centers of circumcircles of triangles AMC, AMB, BMC are P, Q, R, respectively. Prove that the area of $\triangle PQR$ is greater than the area of $\triangle ABC$.

Day 2

5 For an arbitrary real number x, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Prove that there is exactly one integer m which satisfy $m - \left\lfloor \frac{m}{2005} \right\rfloor = 2005$.

6 Find all triples (x, y, z) of integers which satisfy

$x(y+z) = y^2 + z^2 - 2$

$y(z+x) = z^2 + x^2 - 2$

$z(x+y) = x^2 + y^2 - 2.$

7 Let $ABCD$ be a convex quadrilateral. Square AB_1A_2B is constructed such that the two vertices A_2, B_1 is located outside $ABCD$. Similarly, we construct squares BC_1B_2C, CD_1C_2D, DA_1D_2A. Let K be the intersection of AA_2 and BB_1, L be the intersection of BB_2 and CC_1, M be the intersection of CC_2 and DD_1, and N be the intersection of DD_2 and AA_1. Prove that KM is perpendicular to LN.

8 There are 90 contestants in a mathematics competition. Each contestant gets acquainted with at least 60 other contestants. One of the contestants, Amin, state that at least four contestants have the same number of new friends. Prove or disprove his statement.