National Science Olympiad 2006

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Day 1

1 Find all pairs (x, y) of real numbers which satisfy $x^3 - y^3 = 4(x - y)$ and $x^3 + y^3 = 2(x + y)$.

2 Let a, b, c be positive integers. If $30|a + b + c$, prove that $30|a^5 + b^5 + c^5$.

3 Let S be the set of all triangles ABC which have property: $\tan A, \tan B, \tan C$ are positive integers. Prove that all triangles in S are similar.

4 A black pawn and a white pawn are placed on the first square and the last square of a $1 \times n$ chessboard, respectively. Wiwit and Siti move alternatingly. Wiwit has the white pawn, and Siti has the black pawn. The white pawn moves first. In every move, the player moves her pawn one or two squares to the right or to the left, without passing the opponent's pawn. The player who cannot move anymore loses the game. Which player has the winning strategy? Explain the strategy.

Day 2

5 In triangle ABC, M is the midpoint of side BC and G is the centroid of triangle ABC. A line l passes through G, intersecting line AB at P and line AC at Q, where $P \neq B$ and $Q \neq C$. If $[XYZ]$ denotes the area of triangle XYZ, show that $\frac{[BGM]}{[PAG]} + \frac{[CMG]}{[QGA]} = \frac{3}{2}$.

6 Every phone number in an area consists of eight digits and starts with digit 8. Mr Edy, who has just moved to the area, apply for a new phone number. What is the chance that Mr Edy gets a phone number which consists of at most five different digits?

7 Let a, b, c be real numbers such that ab, bc, ca are rational numbers. Prove that there are integers x, y, z, not all of them are 0, such that $ax + by + cz = 0$.

8 Find the largest 85-digit integer which has property: the sum of its digits equals to the product of its digits.