National Science Olympiad 2007

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Day 1

1 Let ABC be a triangle with $\angle ABC = \angle ACB = 70^\circ$. Let point D on side BC such that AD is the altitude, point E on side AB such that $\angle ACE = 10^\circ$, and point F is the intersection of AD and CE. Prove that $CF = BC$.

2 For every positive integer n, $b(n)$ denote the number of positive divisors of n and $p(n)$ denote the sum of all positive divisors of n. For example, $b(14) = 4$ and $p(14) = 24$. Let k be a positive integer greater than 1.

(a) Prove that there are infinitely many positive integers n which satisfy $b(n) = k^2 - k + 1$.

(b) Prove that there are finitely many positive integers n which satisfy $p(n) = k^2 - k + 1$.

3 Let a, b, c be positive real numbers which satisfy $5(a^2 + b^2 + c^2) < 6(ab + bc + ca)$. Prove that these three inequalities hold: $a + b > c$, $b + c > a$, $c + a > b$.

4 A 10-digit arrangement $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ is called *beautiful* if (i) when read left to right, $0, 1, 2, 3, 4$ form an increasing sequence, and $5, 6, 7, 8, 9$ form a decreasing sequence, and (ii) 0 is not the leftmost digit. For example, 9807123654 is a beautiful arrangement. Determine the number of beautiful arrangements.

Day 2

5 Let r, s be two positive integers and P a 'chessboard' with r rows and s columns. Let M denote the maximum value of rooks placed on P such that no two of them attack each other.

(a) Determine M.

(b) How many ways to place M rooks on P such that no two of them attack each other?

[Note: In chess, a rook moves and attacks in a straight line, horizontally or vertically.]

6 Find all triples (x, y, z) of real numbers which satisfy the simultaneous equations

$$x = y^3 + y - 8$$
$$y = z^3 + z - 8$$

$$z = x^3 + x - 8.$$

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7 Points A, B, C, D are on circle S, such that AB is the diameter of S, but CD is not the diameter. Given also that C and D are on different sides of AB. The tangents of S at C and D intersect at P. Points Q and R are the intersections of line AC with line BD and line AD with line BC, respectively.

(a) Prove that P, Q, and R are collinear.

(b) Prove that QR is perpendicular to line AB.

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8 Let m and n be two positive integers. If there are infinitely many integers k such that $k^2+2kn+m^2$ is a perfect square, prove that $m = n$.
