**National Science Olympiad 2008**

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by dyd

**Day 1**

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| **1** | Given triangle $ABC$. Points $D, E, F$ outside triangle $ABC$ are chosen such that triangles $ABD$, $BCE$, and $CAF$ are equilateral triangles. Prove that cicumcircles of these three triangles are concurrent. |
| **2** | Prove that for $x, y \in \mathbb{R}^+$, $\frac{1}{(1+\sqrt{x})^2} + \frac{1}{(1+\sqrt{y})^2} \geq \frac{2}{x+y+2}$ |
| **3** | Find all natural number which can be expressed in $\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b}$ where $a, b, c \in \mathbb{N}$ satisfy $\gcd(a,b) = \gcd(b,c) = \gcd(c,a) = 1$ |
| **4** | Let $A = \{1, 2, \ldots, 2008\}$<br>a) Find the number of subset of $A$ which satisfy : the product of its elements is divisible by 7<br>b) Let $N(i)$ denotes the number of subset of $A$ which sum of its elements remains $i$ when divided by 7. Prove that $N(0) - N(1) + N(2) - N(3) + N(4) - N(5) + N(6) - N(7) = 0$<br><br>EDITED : thx for cosinator.. BTW, your statement and my correction give 80 |

**Day 2**

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| **1** | Let $m, n > 1$ are integers which satisfy $n \mid 4^m - 1$ and $2^m \mid n - 1$. Is it a must that $n = 2^m + 1$? |
| **2** | In a group of 21 persons, every two person communicate with different radio frequency. It's possible for two person to not communicate (means there's no frequency occupied to connect them). Only one frequency used by each couple, and it's unique for every couple. In every 3 persons, exactly two of them is not communicating to each other. Determine the maximum number of frequency required for this group. Explain your answer. |
| **3** | Given triangle $ABC$ with sidelengths $a, b, c$. Tangents to incircle of $ABC$ that parallel with triangle's sides form three small triangle (each small triangle has 1 vertex of $ABC$). Prove that the sum of area of incircles of these three small triangles and the area of incircle of triangle $ABC$ is equal to $\frac{\pi(a^2+b^2+c^2)(b+c-a)(c+a-b)(a+b-c)}{(a+b+c)^3}$<br><br>(hmm,, looks familiar, isn't it? :wink: ) |
| **4** | Find all function $f : \mathbb{N} \to \mathbb{N}$ satisfy $f(mn) + f(m+n) = f(m)f(n) + 1$ for all natural number $n$ |