National Science Olympiad 2009

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Day 1

1 Find all positive integers $n \in \{1, 2, 3, \ldots, 2009\}$ such that

$$4n^6 + n^3 + 5$$

is divisible by 7.

2 For any real x, let $\lfloor x \rfloor$ be the largest integer that is not more than x. Given a sequence of positive integers $a_1, a_2, a_3, \ldots$ such that $a_1 > 1$ and

$$\left\lfloor \frac{a_1 + 1}{a_2} \right\rfloor = \left\lfloor \frac{a_2 + 1}{a_3} \right\rfloor = \left\lfloor \frac{a_3 + 1}{a_4} \right\rfloor = \cdots$$

Prove that

$$\left\lfloor \frac{a_n + 1}{a_{n+1}} \right\rfloor \le 1$$

holds for every positive integer n.

3 For every triangle ABC, let D, E, F be a point located on segment BC, CA, AB, respectively. Let P be the intersection of AD and EF. Prove that:

$$\frac{AB}{AF} \times DC + \frac{AC}{AE} \times DB = \frac{AD}{AP} \times BC$$

4 In an island, there exist 7 towns and a railway system which connected some of the towns. Every railway segment connects 2 towns, and in every town there exists at least 3 railway segments that connects the town to another towns. Prove that there exists a route that visits 4 different towns once and go back to the original town. (Example: $A - B - C - D - A$)

Day 2

1 In a drawer, there are at most 2009 balls, some of them are white, the rest are blue, which are randomly distributed. If two balls were taken at the same time, then the probability that the balls are both blue or both white is $\frac{1}{2}$. Determine the maximum amount of white balls in the drawer, such that the probability statement is true?

2 Find the lowest possible values from the function

$$f(x) = x^{2008} - 2x^{2007} + 3x^{2006} - 4x^{2005} + 5x^{2004} - \cdots - 2006x^3 + 2007x^2 - 2008x + 2009$$

for any real numbers x.

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3 A pair of integers (m, n) is called *good* if

$$m \mid n^2 + n \text{ and } n \mid m^2 + m$$

Given 2 positive integers $a, b > 1$ which are relatively prime, prove that there exists a *good* pair (m, n) with $a \mid m$ and $b \mid n$, but $a \nmid n$ and $b \nmid m$.

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4 Given an acute triangle ABC. The incircle of triangle ABC touches BC, CA, AB respectively at D, E, F. The angle bisector of $\angle A$ cuts DE and DF respectively at K and L. Suppose AA_1 is one of the altitudes of triangle ABC, and M be the midpoint of BC.
(a) Prove that BK and CL are perpendicular with the angle bisector of $\angle BAC$.
(b) Show that A_1KML is a cyclic quadrilateral.
