Baltic Way 2016

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by socrates

1 Find all pairs of primes (p, q) such that

$$p^3 - q^5 = (p+q)^2.$$

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2 Prove or disprove the following hypotheses.
a) For all $k \geq 2,$ each sequence of k consecutive positive integers contains a number that is not divisible by any prime number less than $k.$
b) For all $k \geq 2,$ each sequence of k consecutive positive integers contains a number that is relatively prime to all other members of the sequence.

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3 For which integers $n = 1, \dots, 6$ does the equation

$$a^n + b^n = c^n + n$$

have a solution in integers?

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4 Let n be a positive integer and let a, b, c, d be integers such that $n|a+b+c+d$ and $n|a^2+b^2+c^2+d^2.$
Show that

$$n|a^4+b^4+c^4+d^4+4abcd.$$

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5 Let $p > 3$ be a prime such that $p \equiv 3 \pmod 4.$ Given a positive integer a_0 define the sequence $a_0, a_1, \dots$ of integers by $a_n = a_{n-1}^{2^n}$ for all $n = 1, 2, \dots.$ Prove that it is possible to choose a_0 such that the subsequence $a_N, a_{N+1}, a_{N+2}, \dots$ is not constant modulo p for any positive integer $N.$

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6 The set $\{1, 2, ..., 10\}$ is partitioned to three subsets A, B and $C.$ For each subset the sum of its elements, the product of its elements and the sum of the digits of all its elements are calculated.
Is it possible that A alone has the largest sum of elements, B alone has the largest product of elements, and C alone has the largest sum of digits?

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7 Find all positive integers n for which

$$3x^n + n(x+2) - 3 \geq nx^2$$

holds for all real numbers $x.$

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8 Find all real numbers a for which there exists a non-constant function $f:\mathbb{R}\to\mathbb{R}$ satisfying the following two equations for all $x\in\mathbb{R}$:
i) $f(ax)=a^2f(x)$ and
ii) $f(f(x))=af(x)$.

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9 Find all quadruples (a,b,c,d) of real numbers that simultaneously satisfy the following equations:

$$\begin{cases} a^3+c^3=2 \\ a^2b+c^2d=0 \\ b^3+d^3=1 \\ ab^2+cd^2=-6. \end{cases}$$

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10 Let $a_{0,1},a_{0,2},...,a_{0,2016}$ be positive real numbers. For $n\geq 0$ and $1\leq k<2016$ set

$$a_{n+1,k}=a_{n,k}+\frac{1}{2a_{n,k+1}} \text{ and } a_{n+1,2016}=a_{n,2016}+\frac{1}{2a_{n,1}}.$$

Show that $\max_{1\leq k\leq 2016}a_{2016,k}>44.$

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11 Set A consists of 2016 positive integers. All prime divisors of these numbers are smaller than $30.$ Prove that there are four distinct numbers a,b,c and d in A such that $abcd$ is a perfect square.

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12 Does there exist a hexagon (not necessarily convex) with side lengths $1,2,3,4,5,6$ (not necessarily in this order) that can be tiled with a) 31 b) 32 equilateral triangles with side length $1?$

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13 Let n numbers all equal to 1 be written on a blackboard. A move consists of replacing two numbers on the board with two copies of their sum. It happens that after h moves all n numbers on the blackboard are equal to $m.$ Prove that $h\leq \frac{1}{2}n\log_2 m.$

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14 A cube consists of 4^3 unit cubes each containing an integer. At each move, you choose a unit cube and increase by 1 all the integers in the neighbouring cubes having a face in common with the chosen cube. Is it possible to reach a position where all the 4^3 integers are divisible by $3,$ no matter what the starting position is?

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15 The Baltic Sea has 2016 harbours. There are two-way ferry connections between some of them. It is impossible to make a sequence of direct voyages $C_1-C_2-...-C_{1062}$ where all the harbours $C_1,...,C_{1062}$ are distinct. Prove that there exist two disjoint sets A and B of 477 harbours each, such that there is no harbour in A with a direct ferry connection to a harbour in $B.$

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16 In triangle ABC, the points D and E are the intersections of the angular bisectors from C and B with the sides AB and AC, respectively. Points F and G on the extensions of AB and AC beyond B and C, respectively, satisfy $BF = CG = BC$. Prove that $FG \parallel DE$.

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17 Let $ABCD$ be a convex quadrilateral with $AB = AD$. Let T be a point on the diagonal AC such that $\angle ABT + \angle ADT = \angle BCD$. Prove that $AT + AC \geq AB + AD$.

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18 Let $ABCD$ be a parallelogram such that $\angle BAD = 60^\circ$. Let K and L be the midpoints of BC and CD, respectively. Assuming that $ABKL$ is a cyclic quadrilateral, find $\angle ABD$.

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19 Consider triangles in the plane where each vertex has integer coordinates. Such a triangle can be *legally transformed* by moving one vertex parallel to the opposite side to a different point with integer coordinates. Show that if two triangles have the same area, then there exists a series of legal transformations that transforms one to the other.

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20 Let $ABCD$ be a cyclic quadrilateral with AB and CD not parallel. Let M be the midpoint of CD. Let P be a point inside $ABCD$ such that $PA = PB = CM$. Prove that AB, CD and the perpendicular bisector of MP are concurrent.
