National Science Olympiad 2012

www.artofproblemsolving.com/community/c3653

by nivotko, delegat

Day 1

---

1 Show that for any positive integers a and b, the number

$$n = \text{LCM}(a,b) + \text{GCD}(a,b) - a - b$$

is an even non-negative integer.

Proposer: Nanang Susyanto

---

2 Let $n \ge 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n.$$

Proposed by Angelo Di Pasquale, Australia

---

3 Given an acute triangle ABC with $AB > AC$ that has circumcenter O. Line BO and CO meet the bisector of $\angle BAC$ at P and Q, respectively. Moreover, line BQ and CP meet at R. Show that AR is perpendicular to BC.

Proposer: Soewono and Fajar Yuliawan

---

4 Given 2012 distinct points $A_1, A_2, \ldots, A_{2012}$ on the Cartesian plane. For any permutation $B_1, B_2, \ldots, B_{2012}$ of $A_1, A_2, \ldots, A_{2012}$ define the *shadow* of a point P as follows: [i]Point P is rotated by 180° around B_1 resulting P_1, point P_1 is rotated by 180° around B_2 resulting P_2, ..., point P_{2011} is rotated by 180° around B_{2012} resulting P_{2012}. Then, P_{2012} is called the shadow of P with respect to the permutation $B_1, B_2, \ldots, B_{2012}$.[/i]
Let N be the number of different shadows of P up to all permutations of $A_1, A_2, \ldots, A_{2012}$. Determine the maximum value of N.

Proposer: Hendrata Dharmawan

Day 2

---

1 Given positive integers m and n. Let P and Q be two collections of $m \times n$ numbers of 0 and 1, arranged in m rows and n columns. An example of such collections for $m = 3$ and $n = 4$ is

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let those two collections satisfy the following properties:
(i) On each row of P, from left to right, the numbers are non-increasing,
(ii) On each column of Q, from top to bottom, the numbers are non-increasing,
(iii) The sum of numbers on the row in P equals to the same row in Q,
(iv) The sum of numbers on the column in P equals to the same column in Q.
Show that the number on row i and column j of P equals to the number on row i and column j of Q for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Proposer: Stefanus Lie

---

2 Let $\mathbb{R}^+$ be the set of all positive real numbers. Show that there is no function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f(x+y) = f(x) + f(y) + \frac{1}{2012}$$

for all positive real numbers x and y.

Proposer: Fajar Yuliawan

---

3 Let n be a positive integer. Show that the equation

$$\sqrt{x} + \sqrt{y} = \sqrt{n}$$

have solution of pairs of positive integers (x, y) if and only if n is divisible by some perfect square greater than 1.

Proposer: Nanang Susyanto

---

4 Given a triangle ABC, let the bisector of $\angle BAC$ meets the side BC and circumcircle of triangle ABC at D and E, respectively. Let M and N be the midpoints of BD and CE, respectively. Circumcircle of triangle ABD meets AN at Q. Circle passing through A that is tangent to BC at D meets line AM and side AC respectively at P and R. Show that the four points B, P, Q, R lie on the same line.

Proposer: Fajar Yuliawan
