National Science Olympiad 2013

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Day 1

1 In a 4×6 grid, all edges and diagonals are drawn (see attachment). Determine the number of parallelograms in the grid that uses only the line segments drawn and none of its four angles are right.

2 Let ABC be an acute triangle and ω be its circumcircle. The bisector of $\angle BAC$ intersects ω at [another point] M. Let P be a point on AM and inside $\triangle ABC$. Lines passing P that are parallel to AB and AC intersects BC on E, F respectively. Lines ME, MF intersects ω at points K, L respectively. Prove that AM, BL, CK are concurrent.

3 Determine all positive real M such that for any positive reals a, b, c, at least one of $a + \frac{M}{ab}, b + \frac{M}{bc}, c + \frac{M}{ca}$ is greater than or equal to $1 + M$.

4 Suppose $p > 3$ is a prime number and

$$S = \sum_{2 \le i < j < k \le p-1} ijk$$

Prove that $S + 1$ is divisible by p.

Day 2

5 Let P be a quadratic (polynomial of degree two) with a positive leading coefficient and negative discriminant. Prove that there exists three quadratics P_1, P_2, P_3 such that:
- $P(x) = P_1(x) + P_2(x) + P_3(x)$
- P_1, P_2, P_3 have positive leading coefficients and zero discriminants (and hence each has a double root)
- The roots of P_1, P_2, P_3 are different

6 A positive integer n is called "strong" if there exists a positive integer x such that $x^{nx} + 1$ is divisible by 2^n.

a. Prove that 2013 is strong.
b. If m is strong, determine the smallest y (in terms of m) such that $y^{my} + 1$ is divisible by 2^m.

7 Let $ABCD$ be a parallelogram. Construct squares $ABC_1D_1, BCD_2A_2, CDA_3B_3, DAB_4C_4$ on the outer side of the parallelogram. Construct a square having B_4D_1 as one of its sides and it is on the outer side of AB_4D_1 and call its center O_A. Similarly do it for C_1A_2, D_2B_3, A_3C_4 to obtain O_B, O_C, O_D. Prove that $AO_A = BO_B = CO_C = DO_D$.

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8 Let A be a set of positive integers. A is called "balanced" if [and only if] the number of 3-element subsets of A whose elements add up to a multiple of 3 is equal to the number of 3-element subsets of A whose elements add up to not a multiple of 3.

a. Find a 9-element balanced set.
b. Prove that no set of 2013 elements can be balanced.
