National Science Olympiad 2014

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Day 1

1 Is it possible to fill a 3×3 grid with each of the numbers $1, 2, \dots, 9$ once each such that the sum of any two numbers sharing a side is prime?

2 For some positive integers m, n, the system $x + y^2 = m$ and $x^2 + y = n$ has exactly one integral solution (x, y). Determine all possible values of $m - n$.

3 Let $ABCD$ be a trapezoid (quadrilateral with one pair of parallel sides) such that $AB < CD$. Suppose that AC and BD meet at E and AD and BC meet at F. Construct the parallelograms $AEDK$ and $BECL$. Prove that EF passes through the midpoint of the segment KL.

4 Determine all polynomials with integral coefficients $P(x)$ such that if a, b, c are the sides of a right-angled triangle, then $P(a), P(b), P(c)$ are also the sides of a right-angled triangle. (Sides of a triangle are necessarily positive. Note that it's not necessary for the order of sides to be preserved; if c is the hypotenuse of the first triangle, it's not necessary that $P(c)$ is the hypotenuse of the second triangle, and similar with the others.)

Day 2

1 A sequence of positive integers $a_1, a_2, \dots$ satisfies $a_k + a_l = a_m + a_n$ for all positive integers k, l, m, n satisfying $kl = mn$. Prove that if p divides q then $a_p \le a_q$.

2 Let ABC be a triangle. Suppose D is on BC such that AD bisects $\angle BAC$. Suppose M is on AB such that $\angle MDA = \angle ABC$, and N is on AC such that $\angle NDA = \angle ACB$. If AD and MN intersect on P, prove that $AD^3 = AB \cdot AC \cdot AP$.

3 Suppose that k, m, n are positive integers with $k \le n$. Prove that:

$$\sum_{r=0}^{m} \frac{k\binom{m}{r}\binom{n}{k}}{(r+k)\binom{m+n}{r+k}} = 1$$

4 A positive integer is called *beautiful* if it can be represented in the form $\dfrac{x^2 + y^2}{x + y}$ for two distinct positive integers x, y. A positive integer that is not beautiful is *ugly*.

a) Prove that 2014 is a product of a beautiful number and an ugly number.
b) Prove that the product of two ugly numbers is also ugly.