Korea National Olympiad 2016

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– **Day 1**

1 n is a positive integer. The number of solutions of $x^2+2016y^2=2017^n$ is k. Write k with n.

2 A non-isosceles triangle $\triangle ABC$ has its incircle tangent to BC, CA, AB at points D, E, F. Let the incenter be I. Say AD hits the incircle again at G, at let the tangent to the incircle at G hit AC at H. Let $IH \cap AD = K$, and let the foot of the perpendicular from I to AD be L.

Prove that $IE \cdot IK = IC \cdot IL$.

3 Acute triangle $\triangle ABC$ has area S and perimeter L. A point P inside $\triangle ABC$ has $dist(P,BC)=1, dist(P,CA)=1.5, dist(P,AB)=2$. Let $BC \cap AP = D$, $CA \cap BP = E$, $AB \cap CP = F$. Let T be the area of $\triangle DEF$. Prove the following inequality.

$$\left(\frac{AD \cdot BE \cdot CF}{T}\right)^2 > 4L^2 + \left(\frac{AB \cdot BC \cdot CA}{24S}\right)^2$$

4 For a positive integer n, S_n is the set of positive integer n-tuples $(a_1, a_2, \cdots, a_n)$ which satisfies the following.

(i). $a_1 = 1$.

(ii). $a_{i+1} \le a_i + 1$.

For $k \le n$, define N_k as the number of n-tuples $(a_1, a_2, \cdots a_n) \in S_n$ such that $a_k = 1, a_{k+1} = 2$.

Find the sum $N_1 + N_2 + \cdots N_{k-1}$.

– **Day 2**

5 A non-isosceles triangle $\triangle ABC$ has incenter I and the incircle hits BC, CA, AB at D, E, F. Let EF hit the circumcircle of CEI at $P \neq E$. Prove that $\triangle ABC = 2\triangle ABP$.

6 For a positive integer n, there are n positive reals $a_1 \ge a_2 \ge a_3 \cdots \ge a_n$. For all positive reals $b_1, b_2, \cdots b_n$, prove the following inequality.

$$\frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{a_1 + a_2 + \cdots a_n} \le \max\{\frac{b_1}{1}, \frac{b_1+b_2}{2}, \cdots, \frac{b_1+b_2+\cdots+b_n}{n}\}$$

7 Let $N = 2^a p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$. Prove that there are $(b_1+1)(b_2+1)\dots(b_k+1)$ number of ns which satisfies these two conditions. $\frac{n(n+1)}{2} \le N$, $N - \frac{n(n+1)}{2}$ is divided by n.

8 A subset $S \in \{0, 1, 2, \cdots, 2000\}$ satisfies $|S| = 401$.

Prove that there exists a positive integer n such that there are at least 70 positive integers x such that $x, x+n \in S$