Bosnia Herzegovina Team Selection Test 2009

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Day 1

1 Denote by M and N feets of perpendiculars from A to angle bisectors of exterior angles at B and C, in triangle $\triangle ABC$. Prove that the length of segment MN is equal to semiperimeter of triangle $\triangle ABC$.

2 Find all pairs (a, b) of posive integers such that $\frac{a^2(b-a)}{b+a}$ is square of prime.

3 $a_1, a_2, \dots, a_{100}$ are real numbers such that:

$$a_1 \geq a_2 \geq \dots \geq a_{100} \geq 0$$

$$a_1^2 + a_2^2 \geq 100$$

$$a_3^2 + a_4^2 + \dots + a_{100}^2 \geq 100$$

What is the minimum value of sum $a_1 + a_2 + \dots + a_{100}$.

Day 2

1 Given an 1 x n table ($n \geq 2$), two players alternate the moves in which they write the signs + and - in the cells of the table. The first player always writes +, while the second always writes -. It is not allowed for two equal signs to appear in the adjacent cells. The player who cant make a move loses the game. Which of the players has a winning strategy?

2 Line p intersects sides AB and BC of triangle $\triangle ABC$ at points M and K. If area of triangle $\triangle MBK$ is equal to area of quadrilateral $AMKC$, prove that

$$\frac{|MB| + |BK|}{|AM| + |CA| + |KC|} \geq \frac{1}{3}$$

3 Let n be a positive integer and x positive real number such that none of numbers $x, 2x, \dots, nx$ and none of $\frac{1}{x}, \frac{2}{x}, \dots, \frac{\lfloor nx \rfloor}{x}$ is an integer. Prove that

$$\lfloor x \rfloor + \lfloor 2x \rfloor + \dots + \lfloor nx \rfloor + \left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{2}{x} \right\rfloor + \dots + \left\lfloor \frac{\lfloor nx \rfloor}{x} \right\rfloor = n \lfloor nx \rfloor$$