China Girls Math Olympiad 2002

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Day 1

1 Find all positive integers n such $20n+2$ can divide $2003n+2002$.

2 There are $3n, n \in \mathbb{Z}^+$ girl students who took part in a summer camp. There were three girl students to be on duty every day. When the summer camp ended, it was found that any two of the $3n$ students had just one time to be on duty on the same day.

(1) When $n=3$, is there any arrangement satisfying the requirement above. Prove yor conclusion.

(2) Prove that n is an odd number.

3 Find all positive integers k such that for any positive numbers a, b and c satisfying the inequality

$$k(ab+bc+ca) > 5(a^2+b^2+c^2),$$

there must exist a triangle with a, b and c as the length of its three sides respectively.

4 Circles O_1 and O_2 interest at two points B and C, and BC is the diameter of circle O_1. Construct a tangent line of circle O_1 at C and intersecting circle O_2 at another point A. We join AB to intersect circle O_1 at point E, then join CE and extend it to intersect circle O_2 at point F. Assume H is an arbitrary point on line segment AF. We join HE and extend it to intersect circle O_1 at point G, and then join BG and extend it to intersect the extend line of AC at point D. Prove that

$$\frac{AH}{HF} = \frac{AC}{CD}.$$

Day 2

5 There are $n \geq 2$ permutations $P_1, P_2, \ldots, P_n$ each being an arbitrary permutation of $\{1, \ldots, n\}$. Prove that

$$\sum_{i=1}^{n-1} \frac{1}{P_i + P_{i+1}} > \frac{n-1}{n+2}.$$

6 Find all pairs of positive integers (x, y) such that

$$x^y = y^{x-y}.$$

Albania

7 An acute triangle ABC has three heights AD, BE and CF respectively. Prove that the perimeter of triangle DEF is not over half of the perimeter of triangle ABC.

8 Assume that $A_1, A_2, \ldots, A_8$ are eight points taken arbitrarily on a plane. For a directed line l taken arbitrarily on the plane, assume that projections of $A_1, A_2, \ldots, A_8$ on the line are $P_1, P_2, \ldots, P_8$ respectively. If the eight projections are pairwise disjoint, they can be arranged as $P_{i_1}, P_{i_2}, \ldots, P_{i_8}$ according to the direction of line l. Thus we get one permutation for $1, 2, \ldots, 8$, namely, $i_1, i_2, \ldots, i_8$. In the figure, this permutation is $2, 1, 8, 3, 7, 4, 6, 5$. Assume that after these eight points are projected to every directed line on the plane, we get the number of different permutations as $N_8 = N(A_1, A_2, \ldots, A_8)$. Find the maximal value of N_8.