China Girls Math Olympiad 2003

www.artofproblemsolving.com/community/c3668

by orl

Day 1

---

1 Let ABC be a triangle. Points D and E are on sides AB and AC, respectively, and point F is on line segment DE. Let $\frac{AD}{AB} = x$, $\frac{AE}{AC} = y$, $\frac{DF}{DE} = z$. Prove that

(1) $S_{\triangle BDF} = (1-x)yS_{\triangle ABC}$ and $S_{\triangle CEF} = x(1-y)(1-z)S_{\triangle ABC}$;

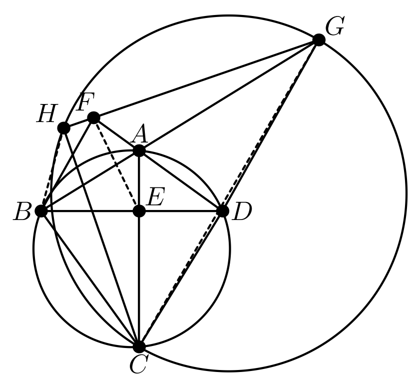
(2) $\sqrt[3]{S_{\triangle BDF}} + \sqrt[3]{S_{\triangle CEF}} \leq \sqrt[3]{S_{\triangle ABC}}$.

---

2 There are 47 students in a classroom with seats arranged in 6 rows $\times$ 8 columns, and the seat in the i-th row and j-th column is denoted by (i,j). Now, an adjustment is made for students seats in the new school term. For a student with the original seat (i,j), if his/her new seat is (m,n), we say that the student is moved by $[a,b] = [i-m, j-n]$ and define the position value of the student as $a+b$. Let S denote the sum of the position values of all the students. Determine the difference between the greatest and smallest possible values of S.

---

3 As shown in the figure, quadrilateral $ABCD$ is inscribed in a circle with AC as its diameter, $BD \perp AC$, and E the intersection of AC and BD. Extend line segment DA and BA through A to F and G respectively, such that $DG||BF$. Extend GF to H such that $CH \perp GH$. Prove that points B, E, F and H lie on one circle.



---

4 (1) Prove that there exist five nonnegative real numbers a, b, c, d and e with their sum equal to 1 such that for any arrangement of these numbers around a circle, there are always two neighboring numbers with their product not less than $\frac{1}{9}$.

(2) Prove that for any five nonnegative real numbers with their sum equal to 1 , it is always possible to arrange them around a circle such that there are two neighboring numbers with their product not greater than $\frac{1}{9}$.

Day 2

5 Let $\{a_n\}_1^\infty$ be a sequence of real numbers such that $a_1 = 2$, and

$$a_{n+1} = a_n^2 - a_n + 1, \forall n \in \mathbb{N}.$$

Prove that

$$1 - \frac{1}{2003^{2003}} < \sum_{i=1}^{2003} \frac{1}{a_i} < 1.$$

6 Let $n \geq 2$ be an integer. Find the largest real number λ such that the inequality

$$a_n^2 \geq \lambda \sum_{i=1}^{n-1} a_i + 2 \cdot a_n.$$

holds for any positive integers $a_1, a_2, \dots a_n$ satisfying $a_1 < a_2 < \dots < a_n$.

7 Let the sides of a scalene triangle $\triangle ABC$ be $AB = c$, $BC = a$, $CA = b$, and D, E, F be points on BC, CA, AB such that AD, BE, CF are angle bisectors of the triangle, respectively. Assume that $DE = DF$. Prove that

(1) $\frac{a}{b+c} = \frac{b}{c+a} + \frac{c}{a+b}$

(2) $\angle BAC > 90°$.

8 Let n be a positive integer, and S_n, be the set of all positive integer divisors of n (including 1 and itself). Prove that at most half of the elements in S_n have their last digits equal to 3.