An Olympiad for grades 7-12.

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P1 Solve the following system (over the real numbers):

$$\begin{cases} 5x+5y+5xy-2xy^2-2x^2y=20 \\ 3x+3y+3xy+xy^2+x^2y=23 \end{cases}$$

P2 A positive integer x satisfies the following:

$$\{\frac{x}{3}\}+\{\frac{x}{5}\}+\{\frac{x}{7}\}+\{\frac{x}{11}\}=\frac{248}{165}$$

Find all possible values of

$$\{\frac{2x}{3}\}+\{\frac{2x}{5}\}+\{\frac{2x}{7}\}+\{\frac{2x}{11}\}$$

where $\{y\}$ denotes the fractional part of y.

P3 A triangle is composed of circular cells arranged in 5784 rows: the first row has one cell, the second has two cells, and so on (see the picture). The cells are divided into pairs of adjacent cells (circles touching each other), so that each cell belongs to exactly one pair. A pair of adjacent cells is called **diagonal** if the two cells in it *aren't* in the same row. What is the minimum possible amount of diagonal pairs in the division?
An example division into pairs is depicted in the image.

P4 Acute triangle ABC is inscribed in a circle with center O. The reflections of O across the three altitudes of the triangle are called U, V, W: U over the altitude from A, V over the altitude from B, and W over the altitude from C.
Let ℓ_A be a line through A parallel to VW, and define ℓ_B, ℓ_C similarly. Prove that the three lines ℓ_A, ℓ_B, ℓ_C are concurrent.

P5 For positive integral $k>1$, we let $p(k)$ be its smallest prime divisor. Given an integer $a_1>2$, we define an infinite sequence a_n by $a_{n+1}=a_n^n-1$ for each $n\geq 1$. For which values of a_1 is the sequence $p(a_n)$ bounded?

P6 Quadrilateral $ABCD$ is inscribed in a circle. Let $\omega_A, \omega_B, \omega_C, \omega_D$ be the incircles of triangles DAB, ABC, BCD, CDA respectively. The common external common tangent of ω_A, ω_B, different from line AB, meets the external common tangent of ω_A, ω_D, different from AD, at point A'. Similarly,

the external common tangent of ω_B, ω_C different from BC meets the external common tangent of ω_C, ω_D different from CD at C'.
Prove that $AA' \parallel CC'$.

P7 A rook stands in one cell of an infinite square grid. A different cell was colored blue and mines were placed in n additional cells: the rook cannot stand on or pass through them. It is known that the rook can reach the blue cell in finitely many moves. Can it do so (for every n and such a choice of mines, starting point, and blue cell) in at most
(a) $1.99n + 100$ moves?
(b) $2n - 2\sqrt{3n} + 100$ moves?

Remark. In each move, the rook goes in a vertical or horizontal line.
