**Sharygin Geometry Olympiad 2009**

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by Snakes, April, parmenides51

– **First (Correspondence) Round**

1 Points $B_1$ and $B_2$ lie on ray $AM$, and points $C_1$ and $C_2$ lie on ray $AK$. The circle with center $O$ is inscribed into triangles $AB_1C_1$ and $AB_2C_2$. Prove that the angles $B_1OB_2$ and $C_1OC_2$ are equal.

2 Given nonisosceles triangle $ABC$. Consider three segments passing through different vertices of this triangle and bisecting its perimeter. Are the lengths of these segments certainly different?

3 The bisectors of trapezoid's angles form a quadrilateral with perpendicular diagonals. Prove that this trapezoid is isosceles.

4 Let $P$ and $Q$ be the common points of two circles. The ray with origin $Q$ reflects from the first circle in points $A_1$, $A_2$,... according to the rule "the angle of incidence is equal to the angle of reflection". Another ray with origin $Q$ reflects from the second circle in the points $B_1$, $B_2$,... in the same manner. Points $A_1$, $B_1$ and $P$ occurred to be collinear. Prove that all lines $A_iB_i$ pass through P.

5 Given triangle $ABC$. Point $O$ is the center of the excircle touching the side $BC$. Point $O_1$ is the reflection of $O$ in $BC$. Determine angle $A$ if $O_1$ lies on the circumcircle of $ABC$.

6 Find the locus of excenters of right triangles with given hypotenuse.

7 Given triangle $ABC$. Points $M$, $N$ are the projections of $B$ and $C$ to the bisectors of angles $C$ and $B$ respectively. Prove that line $MN$ intersects sides $AC$ and $AB$ in their points of contact with the incircle of $ABC$.

8 Some polygon can be divided into two equal parts by three different ways. Is it certainly valid that this polygon has an axis or a center of symmetry?

9 Given $n$ points on the plane, which are the vertices of a convex polygon, $n > 3$. There exists $k$ regular triangles with the side equal to $1$ and the vertices at the given points.
- Prove that $k < \frac{2}{3}n$. - Construct the configuration with $k > 0.666n$.

10 Let $ABC$ be an acute triangle, $CC_1$ its bisector, $O$ its circumcenter. The perpendicular from $C$ to $AB$ meets line $OC_1$ in a point lying on the circumcircle of $AOB$. Determine angle $C$.

**11** Given quadrilateral $ABCD$. The circumcircle of $ABC$ is tangent to side $CD$, and the circumcircle of $ACD$ is tangent to side $AB$. Prove that the length of diagonal $AC$ is less than the distance between the midpoints of $AB$ and $CD$.

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**12** Let $CL$ be a bisector of triangle $ABC$. Points $A_1$ and $B_1$ are the reflections of $A$ and $B$ in $CL$, points $A_2$ and $B_2$ are the reflections of $A$ and $B$ in $L$. Let $O_1$ and $O_2$ be the circumcenters of triangles $AB_1B_2$ and $BA_1A_2$ respectively. Prove that angles $O_1CA$ and $O_2CB$ are equal.

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**13** In triangle $ABC$, one has marked the incenter, the foot of altitude from vertex $C$ and the center of the excircle tangent to side $AB$. After this, the triangle was erased. Restore it.

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**14** Given triangle $ABC$ of area 1. Let $BM$ be the perpendicular from $B$ to the bisector of angle $C$. Determine the area of triangle $AMC$.

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**15** Given a circle and a point $C$ not lying on this circle. Consider all triangles $ABC$ such that points $A$ and $B$ lie on the given circle. Prove that the triangle of maximal area is isosceles.

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**16** Three lines passing through point $O$ form equal angles by pairs. Points $A_1$, $A_2$ on the first line and $B_1$, $B_2$ on the second line are such that the common point $C_1$ of $A_1B_1$ and $A_2B_2$ lies on the third line. Let $C_2$ be the common point of $A_1B_2$ and $A_2B_1$. Prove that angle $C_1OC_2$ is right.

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**17** Given triangle $ABC$ and two points $X$, $Y$ not lying on its circumcircle. Let $A_1$, $B_1$, $C_1$ be the projections of $X$ to $BC$, $CA$, $AB$, and $A_2$, $B_2$, $C_2$ be the projections of $Y$. Prove that the perpendiculars from $A_1$, $B_1$, $C_1$ to $B_2C_2$, $C_2A_2$, $A_2B_2$, respectively, concur if and only if line $XY$ passes through the circumcenter of $ABC$.

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**18** Given three parallel lines on the plane. Find the locus of incenters of triangles with vertices lying on these lines (a single vertex on each line).

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**19** Given convex $n$-gon $A_1 \dots A_n$. Let $P_i$ ($i = 1, \dots, n$) be such points on its boundary that $A_iP_i$ bisects the area of polygon. All points $P_i$ don't coincide with any vertex and lie on $k$ sides of $n$-gon. What is the maximal and the minimal value of $k$ for each given $n$?

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**20** Suppose $H$ and $O$ are the orthocenter and the circumcenter of acute triangle $ABC$; $AA_1$, $BB_1$ and $CC_1$ are the altitudes of the triangle. Point $C_2$ is the reflection of $C$ in $A_1B_1$. Prove that $H$, $O$, $C_1$ and $C_2$ are concyclic.

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**21** The opposite sidelines of quadrilateral $ABCD$ intersect at points $P$ and $Q$. Two lines passing through these points meet the side of $ABCD$ in four points which are the vertices of a parallelogram. Prove that the center of this parallelogram lies on the line passing through the midpoints of diagonals of $ABCD$.

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**22** Construct a quadrilateral which is inscribed and circumscribed, given the radii of the respective circles and the angle between the diagonals of quadrilateral.

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**23** Is it true that for each $n$, the regular $2n$-gon is a projection of some polyhedron having not greater than $n+2$ faces?

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**24** A sphere is inscribed into a quadrangular pyramid. The point of contact of the sphere with the base of the pyramid is projected to the edges of the base. Prove that these projections are concyclic.

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**–** **Final Round**

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**–** Grade 8

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**1** Minor base $BC$ of trapezoid $ABCD$ is equal to side $AB$, and diagonal $AC$ is equal to base $AD$. The line passing through B and parallel to $AC$ intersects line $DC$ in point $M$. Prove that $AM$ is the bisector of angle $\angle BAC$.

A.Blinkov, Y.Blinkov

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**2** A cyclic quadrilateral is divided into four quadrilaterals by two lines passing through its inner point. Three of these quadrilaterals are cyclic with equal circumradii. Prove that the fourth part also is cyclic quadrilateral and its circumradius is the same.

(A.Blinkov)

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**3** Let $AH_a$ and $BH_b$ be the altitudes of triangle $ABC$. Points $P$ and $Q$ are the projections of $H_a$ to $AB$ and $AC$. Prove that line $PQ$ bisects segment $H_aH_b$.

(A.Akopjan, K.Savenkov)

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**4** Given is $\triangle ABC$ such that $\angle A = 57^o, \angle B = 61^o$ and $\angle C = 62^o$. Which segment is longer: the angle bisector through $A$ or the median through $B$?

(N.Beluhov)

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**5** Given triangle $ABC$. Point $M$ is the projection of vertex $B$ to bisector of angle $C$. $K$ is the touching point of the incircle with side $BC$. Find angle $\angle MKB$ if $\angle BAC = \alpha$

(V.Protasov)

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**6** Can four equal polygons be placed on the plane in such a way that any two of them don't have common interior points, but have a common boundary segment?

(S.Markelov)

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**7** Let $s$ be the circumcircle of triangle $ABC, L$ and $W$ be common points of angle's $A$ bisector with side $BC$ and $s$ respectively, $O$ be the circumcenter of triangle $ACL$. Restore triangle $ABC$, if circle $s$ and points $W$ and $O$ are given.

(D.Prokopenko)

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**8** A triangle $ABC$ is given, in which the segment $BC$ touches the incircle and the corresponding excircle in points $M$ and $N$. If $\angle BAC = 2\angle MAN$, show that $BC = 2MN$.

(N.Beluhov)

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**–** Grade 9

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**1** The midpoint of triangle's side and the base of the altitude to this side are symmetric wrt the touching point of this side with the incircle. Prove that this side equals one third of triangle's perimeter.

(A.Blinkov, Y.Blinkov)

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**2** Given a convex quadrilateral $ABCD$. Let $R_a, R_b, R_c$ and $R_d$ be the circumradii of triangles $DAB, ABC, BCD$. Prove that inequality $R_a < R_b < R_c < R_d$ is equivalent to $180^o - \angle CDB < \angle CAB < \angle CDB$ .

(O.Musin)

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**3** Quadrilateral $ABCD$ is circumscribed, rays $BA$ and $CD$ intersect in point $E$, rays $BC$ and $AD$ intersect in point $F$. The incircle of the triangle formed by lines $AB, CD$ and the bisector of angle $B$, touches $AB$ in point $K$, and the incircle of the triangle formed by lines $AD, BC$ and the bisector of angle $B$, touches $BC$ in point $L$. Prove that lines $KL, AC$ and $EF$ concur.

(I.Bogdanov)

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**4** Given regular $17$-gon $A_1 ... A_{17}$. Prove that two triangles formed by lines $A_1A_4, A_2A_{10}, A_{13}A_{14}$ and $A_2A_3, A_4A_6A_{14}A_{15}$ are equal.

(N.Beluhov)

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**5** Let $n$ points lie on the circle. Exactly half of triangles formed by these points are acute-angled. Find all possible $n$.

(B.Frenkin)

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**6** Given triangle $ABC$ such that $AB - BC = \frac{AC}{\sqrt{2}}$ . Let $M$ be the midpoint of $AC$, and $N$ be the foot of the angle bisector from $B$. Prove that $\angle BMC + \angle BNC = 90^o$.

(A.Akopjan)

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**7** Given two intersecting circles with centers $O_1, O_2$. Construct the circle touching one of them externally and the second one internally such that the distance from its center to $O_1O_2$ is maximal.

(M.Volchkevich)

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**8** Given cyclic quadrilateral $ABCD$. Four circles each touching its diagonals and the circumcircle internally are equal. Is $ABCD$ a square?

(C.Pohoata, A.Zaslavsky)

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**–** Grade 10

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**1** Let $a, b, c$ be the lengths of some triangle's sides, $p, r$ be the semiperimeter and the inradius of triangle. Prove an inequality $\sqrt{\frac{ab(p-c)}{p}} + \sqrt{\frac{ca(p-b)}{p}} + \sqrt{\frac{bc(p-a)}{p}} \geq 6r$

(D.Shvetsov)

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**2** Given quadrilateral $ABCD$. Its sidelines$AB$ and $CD$ intersect in point $K$. It's diagonals intersect in point $L$. It is known that line $KL$ pass through the centroid of $ABCD$. Prove that $ABCD$ is trapezoid.

(F.Nilov)

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**3** The cirumradius and the inradius of triangle $ABC$ are equal to $R$ and $r, O, I$ are the centers of respective circles. External bisector of angle $C$ intersect $AB$ in point $P$. Point $Q$ is the projection of $P$ to line $OI$. Find distance $OQ$.

(A.Zaslavsky, A.Akopjan)

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**4** Three parallel lines $d_a, d_b, d_c$ pass through the vertex of triangle $ABC$. The reflections of $d_a, d_b, d_c$ in $BC, CA, AB$ respectively form triangle $XYZ$. Find the locus of incenters of such triangles.

(C.Pohoata)

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**5** Rhombus $CKLN$ is inscribed into triangle $ABC$ in such way that point $L$ lies on side $AB$, point $N$ lies on side $AC$, point $K$ lies on side $BC$. $O_1, O_2$ and $O$ are the circumcenters of triangles $ACL, BCL$ and $ABC$ respectively. Let $P$ be the common point of circles $ANL$ and $BKL$, distinct from $L$. Prove that points $O_1, O_2, O$ and $P$ are concyclic.

(D.Prokopenko)

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**6** Let $M, I$ be the centroid and the incenter of triangle $ABC$, $A_1$ and $B_1$ be the touching points of the incircle with sides $BC$ and $AC$, $G$ be the common point of lines $AA_1$ and $BB_1$. Prove that angle $\angle CGI$ is right if and only if $GM//AB$.

 

(A.Zaslavsky)

**7** Given points $O, A_1, A_2, ..., A_n$ on the plane. For any two of these points the square of distance between them is natural number. Prove that there exist two vectors $\vec{x}$ and $\vec{y}$, such that for any point $A_i$, $\vec{OA_i} = k\vec{x} + l\vec{y}$, where $k$ and $l$ are some integer numbers.

(A.Glazyrin)

**8** Can the regular octahedron be inscribed into regular dodecahedron in such way that all vertices of octahedron be the vertices of dodecahedron?

(B.Frenkin)