**Sharygin Geometry Olympiad 2010**

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– First Round

**1** Does there exist a triangle, whose side is equal to some of its altitudes, another side is equal to some of its bisectors, and the third is equal to some of its medians?

**2** Bisectors $AA_1$ and $BB_1$ of a right triangle $ABC$ $(\angle C = 90^\circ)$ meet at a point $I$. Let $O$ be the circumcenter of triangle $CA_1B_1$. Prove that $OI \perp AB$.

**3** Points $A', B', C'$ lie on sides $BC, CA, AB$ of triangle $ABC$. for a point $X$ one has $\angle AXB = \angle A'C'B' + \angle ACB$ and $\angle BXC = \angle B'A'C' + \angle BAC$. Prove that the quadrilateral $XA'BC'$ is cyclic.

**4** The diagonals of a cyclic quadrilateral $ABCD$ meet in a point $N$. The circumcircles of triangles $ANB$ and $CND$ intersect the sidelines $BC$ and $AD$ for the second time in points $A_1, B_1, C_1, D_1$. Prove that the quadrilateral $A_1B_1C_1D_1$ is inscribed in a circle centered at $N$.

**5** A point $E$ lies on the altitude $BD$ of triangle $ABC$, and $\angle AEC = 90^\circ$. Points $O_1$ and $O_2$ are the circumcenters of triangles $AEB$ and $CEB$; points $F, L$ are the midpoints of the segments $AC$ and $O_1O_2$. Prove that the points $L, E, F$ are collinear.

**6** Points $M$ and $N$ lie on the side $BC$ of the regular triangle $ABC$ ($M$ is between $B$ and $N$), and $\angle MAN = 30^\circ$. The circumcircles of triangles $AMC$ and $ANB$ meet at a point $K$. Prove that the line $AK$ passes through the circumcenter of triangle $AMN$.

**7** The line passing through the vertex $B$ of a triangle $ABC$ and perpendicular to its median $BM$ intersects the altitudes dropped from $A$ and $C$ (or their extensions) in points $K$ and $N$. Points $O_1$ and $O_2$ are the circumcenters of the triangles $ABK$ and $CBN$ respectively. Prove that $O_1M = O_2M$.

**8** Let $AH$ be the altitude of a given triangle $ABC$. The points $I_b$ and $I_c$ are the incenters of the triangles $ABH$ and $ACH$ respectively. $BC$ touches the incircle of the triangle $ABC$ at a point $L$. Find $\angle LI_bI_c$.

**9** A point inside a triangle is called "*good*" if three cevians passing through it are equal. Assume for an isosceles triangle $ABC$ $(AB = BC)$ the total number of "*good*" points is odd. Find all possible values of this number.

**10** Let three lines forming a triangle $ABC$ be given. Using a two-sided ruler and drawing at most eight lines construct a point $D$ on the side $AB$ such that $\frac{AD}{BD} = \frac{BC}{AC}$.

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**11** A convex $n-$gon is split into three convex polygons. One of them has $n$ sides, the second one has more than $n$ sides, the third one has less than $n$ sides. Find all possible values of $n$.

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**12** Let $AC$ be the greatest leg of a right triangle $ABC$, and $CH$ be the altitude to its hypotenuse. The circle of radius $CH$ centered at $H$ intersects $AC$ in point $M$. Let a point $B'$ be the reflection of $B$ with respect to the point $H$. The perpendicular to $AB$ erected at $B'$ meets the circle in a point $K$. Prove that

**a)** $B'M \parallel BC$

**b)** $AK$ is tangent to the circle.

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**13** Let us have a convex quadrilateral $ABCD$ such that $AB = BC$. A point $K$ lies on the diagonal $BD$, and $\angle AKB + \angle BKC = \angle A + \angle C$. Prove that $AK \cdot CD = KC \cdot AD$.

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**14** We have a convex quadrilateral $ABCD$ and a point $M$ on its side $AD$ such that $CM$ and $BM$ are parallel to $AB$ and $CD$ respectively. Prove that $S_{ABCD} \geq 3S_{BCM}$.

*Remark.* $S$ denotes the area function.

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**15** Let $AA_1, BB_1$ and $CC_1$ be the altitudes of an acute-angled triangle $ABC$. $AA_1$ meets $B_1C_1$ in a point $K$. The circumcircles of triangles $A_1KC_1$ and $A_1KB_1$ intersect the lines $AB$ and $AC$ for the second time at points $N$ and $L$ respectively. Prove that

**a)** The sum of diameters of these two circles is equal to $BC$,

**b)** $\frac{A_1N}{BB_1} + \frac{A_1L}{CC_1} = 1.$

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**16** A circle touches the sides of an angle with vertex $A$ at points $B$ and $C$. A line passing through $A$ intersects this circle in points $D$ and $E$. A chord $BX$ is parallel to $DE$. Prove that $XC$ passes through the midpoint of the segment $DE$.

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**17** Construct a triangle, if the lengths of the bisectrix and of the altitude from one vertex, and of the median from another vertex are given.

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**18** A point $B$ lies on a chord $AC$ of circle $\omega$. Segments $AB$ and $BC$ are diameters of circles $\omega_1$ and $\omega_2$ centered at $O_1$ and $O_2$ respectively. These circles intersect $\omega$ for the second time in points $D$ and $E$ respectively. The rays $O_1D$ and $O_2E$ meet in a point $F$, and the rays $AD$ and $CE$ do in a point $G$. Prove that the line $FG$ passes through the midpoint of the segment $AC$.

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**19** A quadrilateral $ABCD$ is inscribed into a circle with center $O$. Points $P$ and $Q$ are opposite to $C$ and $D$ respectively. Two tangents drawn to that circle at these points meet the line $AB$ in points

$E$ and $F$. ($A$ is between $E$ and $B$, $B$ is between $A$ and $F$). The line $EO$ meets $AC$ and $BC$ in points $X$ and $Y$ respectively, and the line $FO$ meets $AD$ and $BD$ in points $U$ and $V$ respectively. Prove that $XV = YU$.

**20** The incircle of an acute-angled triangle $ABC$ touches $AB, BC, CA$ at points $C_1, A_1, B_1$ respectively. Points $A_2, B_2$ are the midpoints of the segments $B_1C_1, A_1C_1$ respectively. Let $P$ be a common point of the incircle and the line $CO$, where $O$ is the circumcenter of triangle $ABC$. Let also $A'$ and $B'$ be the second common points of $PA_2$ and $PB_2$ with the incircle. Prove that a common point of $AA'$ and $BB'$ lies on the altitude of the triangle dropped from the vertex $C$.

**21** A given convex quadrilateral $ABCD$ is such that $\angle ABD + \angle ACD > \angle BAC + \angle BDC$. Prove that

$$S_{ABD} + S_{ACD} > S_{BAC} + S_{BDC}.$$

**22** A circle centered at a point $F$ and a parabola with focus $F$ have two common points. Prove that there exist four points $A, B, C, D$ on the circle such that the lines $AB, BC, CD$ and $DA$ touch the parabola.

**23** A cyclic hexagon $ABCDEF$ is such that $AB \cdot CF = 2BC \cdot FA, CD \cdot EB = 2DE \cdot BC$ and $EF \cdot AD = 2FA \cdot DE$. Prove that the lines $AD, BE$ and $CF$ are concurrent.

**24** Let us have a line $\ell$ in the space and a point $A$ not lying on $\ell$. For an arbitrary line $\ell'$ passing through $A$, $XY$ ($Y$ is on $\ell'$) is a common perpendicular to the lines $\ell$ and $\ell'$. Find the locus of points $Y$.

**25** For two different regular icosahedrons it is known that some six of their vertices are vertices of a regular octahedron. Find the ratio of the edges of these icosahedrons.

**–** Final Round

**–** Grade 8

**1** For a nonisosceles triangle $ABC$, consider the altitude from vertex $A$ and two bisectrices from remaining vertices. Prove that the circumcircle of the triangle formed by these three lines touches the bisectrix from vertex $A$.

**2** Two points $A$ and $B$ are given. Find the locus of points $C$ such that triangle $ABC$ can be covered by a circle with radius $1$.

(Arseny Akopyan)

**3** Let $ABCD$ be a convex quadrilateral and $K$ be the common point of rays $AB$ and $DC$. There exists a point $P$ on the bisectrix of angle $AKD$ such that lines $BP$ and $CP$ bisect segments $AC$ and $BD$ respectively. Prove that $AB = CD$.

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**4** Circles $\omega_1$ and $\omega_2$ inscribed into equal angles $X_1OY$ and $YOX_2$ touch lines $OX_1$ and $OX_2$ at points $A_1$ and $A_2$ respectively. Also they touch $OY$ at points $B_1$ and $B_2$. Let $C_1$ be the second common point of $A_1B_2$ and $\omega_1, C_2$ be the second common point of $A_2B_1$ and $\omega_2$. Prove that $C_1C_2$ is the common tangent of two circles.

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**5** Let $AH$, $BL$ and $CM$ be an altitude, a bisectrix and a median in triangle $ABC$. It is known that lines $AH$ and $BL$ are an altitude and a bisectrix of triangle $HLM$. Prove that line $CM$ is a median of this triangle.

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**6** Let $E, F$ be the midpoints of sides $BC, CD$ of square $ABCD$. Lines $AE$ and $BF$ meet at point $P$. Prove that $\angle PDA = \angle AED$.

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**7** Each of two regular polygons $P$ and $Q$ was divided by a line into two parts. One part of $P$ was attached to one part of $Q$ along the dividing line so that the resulting polygon was regular and not congruent to $P$ or $Q$. How many sides can it have?

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**8** Bisectrices $AA_1$ and $BB_1$ of triangle $ABC$ meet in $I$. Segments $A_1I$ and $B_1I$ are the bases of isosceles triangles with opposite vertices $A_2$ and $B_2$ lying on line $AB$. It is known that line $CI$ bisects segment $A_2B_2$. Is it true that triangle $ABC$ is isosceles?

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**–** Grade 9

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**1** For each vertex of triangle $ABC$, the angle between the altitude and the bisectrix from this vertex was found. It occurred that these angle in vertices $A$ and $B$ were equal. Furthermore the angle in vertex $C$ is greater than two remaining angles. Find angle $C$ of the triangle.

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**2** Two intersecting triangles are given. Prove that at least one of their vertices lies inside the circumcircle of the other triangle.

(Here, the triangle is considered the part of the plane bounded by a closed three-part broken line, a point lying on a circle is considered to be lying inside it.)

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**3** Points $X, Y, Z$ lies on a line (in indicated order). Triangles $XAB$, $YBC$, $ZCD$ are regular, the vertices of the first and the third triangle are oriented counterclockwise and the vertices of the second are opposite oriented. Prove that $AC$, $BD$ and $XY$ concur.

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**4** In triangle $ABC$, touching points $A', B'$ of the incircle with $BC, AC$ and common point $G$ of segments $AA'$ and $BB'$ were marked. After this the triangle was erased. Restore it by the ruler and the compass.

**5** The incircle of a right-angled triangle $ABC$ ($\angle ABC = 90^o$) touches $AB, BC, AC$ in points $C_1, A_1, B_1$, respectively. One of the excircles touches the side $BC$ in point $A_2$. Point $A_0$ is the circumcenter or triangle $A_1A_2B_1$, point $C_0$ is defined similarly. Find angle $A_0BC_0$.

**6** An arbitrary line passing through vertex $B$ of triangle $ABC$ meets side $AC$ at point $K$ and the circumcircle in point $M$. Find the locus of circumcenters of triangles $AMK$.

**7** Given triangle $ABC$. Lines $AL_a$ and $AM_a$ are the internal and the external bisectrix of angle $A$. Let $\omega_a$ be the reflection of the circumcircle of $\triangle AL_aM_a$ in the midpoint of $BC$. Circle $\omega_b$ is defined similarly. Prove that $\omega_a$ and $\omega_b$ touch if and only if $\triangle ABC$ is right-angled.

**8** Given is a regular polygon. Volodya wants to mark $k$ points on its perimeter so that any another regular polygon (maybe having a different number of sides) doesn't contain all marked points on its perimeter. Find the minimal $k$ sufficient for any given polygon.

**–** Grade 10

**1** Let $O, I$ be the circumcenter and the incenter of a right-angled triangle, $R, r$ be the radii of respective circles, $J$ be the reflection of the vertex of the right angle in $I$. Find $OJ$.

**2** Each of two equal circles $\omega_1$ and $\omega_2$ passes through the center of the other one. Triangle $ABC$ is inscribed into $\omega_1$, and lines $AC, BC$ touch $\omega_2$ . Prove that $cosA + cosB = 1$.

**3** All sides of a convex polygon were decreased in such a way that they formed a new convex polygon. Is it possible that all diagonals were increased?

**4** Projections of two points to the sidelines of a quadrilateral lie on two concentric circles (projections of each point form a cyclic quadrilateral and the radii of circles are different). Prove that this quadrilateral is a parallelogram.

**5** Let $BH$ be an altitude of a right-angled triangle $ABC$ ($\angle B = 90^o$). The incircle of triangle $ABH$ touches $AB, AH$ in points $H_1, B_1$, the incircle of triangle $CBH$ touches $CB, CH$ in points $H_2, B_2$, point $O$ is the circumcenter of triangle $H_1BH_2$. Prove that $OB_1 = OB_2$.

**6** The incircle of triangle $ABC$ touches its sides in points $A', B', C'$ . It is known that the orthocenters of triangles $ABC$ and $A'B'C'$ coincide. Is triangle $ABC$ regular?

**7** Each of two regular polyhedrons $P$ and $Q$ was divided by the plane into two parts. One part of $P$ was attached to one part of $Q$ along the dividing plane and formed a regular polyhedron not equal to $P$ and $Q$. How many faces can it have?

**8** Triangle $ABC$ is inscribed into circle $k$. Points $A_1, B_1, C_1$ on its sides were marked, after this the triangle was erased. Prove that it can be restored uniquely if and only if $AA_1, BB_1$ and $CC_1$ concur.

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