Spain Mathematical Olympiad 1965

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–	Day 1
1	We consider an equilateral triangle with its circumscribed circle, of center O, and radius 4cm. We rotate the triangle 90 around O. Compute the common area that was covered by the previous position of the triangle and is also covered by the new one.
2	How many numbers of 3 digits have their central digit greater than any of the other two? How many of them have also three different digits?
3	A disk in a record turntable makes 100 revolutions per minute and it plays during 24 minutes and 30 seconds. The recorded line over the disk is a spiral with a diameter that decreases uniformly from 29cm to 11.5cm. Compute the length of the recorded line.
4	Find all the intervals I where any element of the interval $x \in I$ satisfies $$\cos x + \sin x > 1.$$ Do the same computation when x satisfies $$\cos x + |\sin x| > 1.$$
–	Day 2
5	It is well-known that if $\frac{p}{q} = \frac{r}{s}$, both of the expressions are also equal to $\frac{p-r}{q-s}$. Now we write the equality $$\frac{3x-b}{3x-5b} = \frac{3a-4b}{3a-8b}.$$ The previous property shows that both fractions should be equal to $$\frac{3x-b-3a+4b}{3x-5b-3a+8b} = \frac{3x-3a+3b}{3x-3a+3b} = 1.$$ However, the initial fractions given may not be equal to 1. Explain what is going on.
6	We have an empty equilateral triangle with length of a side l. We put the triangle, horizontally, over a sphere of radius r. Clearly, if the triangle is small enough, the triangle is held by the sphere. Which is the distance between any vertex of the triangle and the centre of the sphere (as a function of l and r)?

7 A truncated cone has the bigger base of radius r centimetres and the generatrix makes an angle, with that base, whose tangent equals m. The truncated cone is constructed of a material of density d (g/cm^3) and the smaller base is covered by a special material of density p (g/cm^2). Which is the height of the truncated cone that maximizes the total mass?

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8 Let be γ_1 a circumference of radius r and P an exterior point that is at distance a from the centre of γ_1. We build two tangent lines r, s to γ_1 from P and γ_2 is constructed as a smaller circumference, tangent to both lines and, also, tangent to γ_1. We construct inductively γ_{n+1} as a tangent circumference to γ_n and, also, tangent to r and s. Determine:

a) The radius of γ_2.
b) The radius of γ_n.
c) The sum of the lengths of $\gamma_1, \gamma_2, \gamma_3, ...$
