Sharygin Geometry Olympiad 2011

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–	First Round
1	Does a convex heptagon exist which can be divided into 2011 equal triangles?
2	Let ABC be a triangle with sides $AB = 4$ and $AC = 6$. Point H is the projection of vertex B to the bisector of angle A. Find MH, where M is the midpoint of BC.
3	Let ABC be a triangle with $\angle A = 60^\circ$. The midperpendicular of segment AB meets line AC at point C_1. The midperpendicular of segment AC meets line AB at point B_1. Prove that line B_1C_1 touches the incircle of triangle ABC.
4	Segments AA', BB', and CC' are the bisectrices of triangle ABC. It is known that these lines are also the bisectrices of triangle $A'B'C'$. Is it true that triangle ABC is regular?
5	Given triangle ABC. The midperpendicular of side AB meets one of the remaining sides at point C'. Points A' and B' are defined similarly. Find all triangles ABC such that triangle $A'B'C'$ is regular.
6	Two unit circles ω_1 and ω_2 intersect at points A and B. M is an arbitrary point of ω_1, N is an arbitrary point of ω_2. Two unit circles ω_3 and ω_4 pass through both points M and N. Let C be the second common point of ω_1 and ω_3, and D be the second common point of ω_2 and ω_4. Prove that $ACBD$ is a parallelogram.
7	Points P and Q on sides AB and AC of triangle ABC are such that $PB = QC$. Prove that $PQ < BC$.
8	The incircle of right-angled triangle ABC ($\angle B = 90^o$) touches AB, BC, CA at points C_1, A_1, B_1 respectively. Points A_2, C_2 are the reflections of B_1 in lines BC, AB respectively. Prove that lines A_1A_2 and C_1C_2 meet on the median of triangle ABC.
9	Let H be the orthocenter of triangle ABC. The tangents to the circumcircles of triangles CHB and AHB at point H meet AC at points A_1 and C_1 respectively. Prove that $A_1H = C_1H$.
10	The diagonals of trapezoid $ABCD$ meet at point O. Point M of lateral side CD and points P, Q of bases BC and AD are such that segments MP and MQ are parallel to the diagonals of the trapezoid. Prove that line PQ passes through point O.

11 The excircle of right-angled triangle ABC ($\angle B = 90^o$) touches side BC at point A_1 and touches line AC in point A_2. Line A_1A_2 meets the incircle of ABC for the first time at point A', point C' is defined similarly. Prove that $AC||A'C'$.

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12 Let AP and BQ be the altitudes of acute-angled triangle ABC. Using a compass and a ruler, construct a point M on side AB such that $\angle AQM = \angle BPM$.

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13 a) Find the locus of centroids for triangles whose vertices lie on the sides of a given triangle (each side contains a single vertex).
b) Find the locus of centroids for tetrahedrons whose vertices lie on the faces of a given tetrahedron (each face contains a single vertex).

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14 In triangle ABC, the altitude and the median from vertex A form (together with line BC) a triangle such that the bisectrix of angle A is the median; the altitude and the median from vertex B form (together with line AC) a triangle such that the bisectrix of angle B is the bisectrix. Find the ratio of sides for triangle ABC.

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15 Given a circle with center O and radius equal to 1. AB and AC are the tangents to this circle from point A. Point M on the circle is such that the areas of quadrilaterals $OBMC$ and $ABMC$ are equal. Find MA.

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16 Given are triangle ABC and line ℓ. The reflections of ℓ in AB and AC meet at point A_1. Points B_1, C_1 are defined similarly. Prove that
a) lines AA_1, BB_1, CC_1 concur,
b) their common point lies on the circumcircle of ABC
c) two points constructed in this way for two perpendicular lines are opposite.

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17 a) Does there exist a triangle in which the shortest median is longer that the longest bisectrix?
b) Does there exist a triangle in which the shortest bisectrix is longer that the longest altitude?

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18 On the plane, given are n lines in general position, i.e. any two of them arent parallel and any three of them dont concur. These lines divide the plane into several parts. What is
a) the minimal,
b) the maximal number of these parts that can be angles?

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19 Does there exist a nonisosceles triangle such that the altitude from one vertex, the bisectrix from the second one and the median from the third one are equal?

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20 Quadrilateral $ABCD$ is circumscribed around a circle with center I. Points M and N are the midpoints of diagonals AC and BD. Prove that $ABCD$ is cyclic quadrilateral if and only if $IM : AC = IN : BD$.

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21 On a circle with diameter AC, let B be an arbitrary point distinct from A and C. Points M, N are the midpoints of chords AB, BC, and points P, Q are the midpoints of smaller arcs restricted by these chords. Lines AQ and BC meet at point K, and lines CP and AB meet at point L. Prove that lines MQ, NP and KL concur.

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22 Let CX, CY be the tangents from vertex C of triangle ABC to the circle passing through the midpoints of its sides. Prove that lines XY, AB and the tangent to the circumcircle of ABC at point C concur.

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23 Given are triangle ABC and line ℓ intersecting BC, CA and AB at points A_1, B_1 and C_1 respectively. Point A' is the midpoint of the segment between the projections of A_1 to AB and AC. Points B' and C' are defined similarly.
(a) Prove that A', B' and C' lie on some line ℓ'.
(b) Suppose ℓ passes through the circumcenter of $\triangle ABC$. Prove that in this case ℓ' passes through the center of its nine-points circle.

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24 Given is an acute-angled triangle ABC. On sides BC, CA, AB, find points A', B', C' such that the longest side of triangle $A'B'C'$ is minimal.

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25 Three equal regular tetrahedrons have the common center. Is it possible that all faces of the polyhedron that forms their intersection are equal?

– Final Round

– Grade 8

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1 The diagonals of a trapezoid are perpendicular, and its altitude is equal to the medial line. Prove that this trapezoid is isosceles

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2 Peter made a paper rectangle, put it on an identical rectangle and pasted both rectangles along their perimeters. Then he cut the upper rectangle along one of its diagonals and along the perpendiculars to this diagonal from two remaining
vertices. After this he turned back the obtained triangles in such a way that they, along with the lower rectangle form a new rectangle.
Let this new rectangle be given. Restore the original rectangle using compass and ruler.

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3 The line passing through vertex A of triangle ABC and parallel to BC meets the circumcircle of ABC for the second time at point A_1. Points B_1 and C_1 are defined similarly. Prove that the perpendiculars from A_1, B_1, C_1 to BC, CA, AB respectively concur.

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4 Given the circle of radius 1 and several its chords with the sum of lengths 1. Prove that one can be inscribe a regular hexagon into that circle so that its sides dont intersect those chords.

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5 A line passing through vertex A of regular triangle ABC doesnt intersect segment BC. Points M and N lie on this line, and $AM = AN = AB$ (point B lies inside angle MAC). Prove that the quadrilateral formed by lines AB, AC, BN, CM is cyclic.

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6 Let BB_1 and CC_1 be the altitudes of acute-angled triangle ABC, and A_0 is the midpoint of BC. Lines A_0B_1 and A_0C_1 meet the line passing through A and parallel to BC in points P and Q. Prove that the incenter of triangle PA_0Q lies on the altitude of triangle ABC.

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7 Let a point M not lying on coordinates axes be given. Points Q and P move along Y - and X-axis respectively so that angle PMQ is always right. Find the locus of points symmetric to M wrt PQ.

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8 Using only the ruler, divide the side of a square table into n equal parts.
All lines drawn must lie on the surface of the table.

– Grade 9

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1 Altitudes AA_1 and BB_1 of triangle ABC meet in point H. Line CH meets the semicircle with diameter AB, passing through A_1, B_1, in point D. Segments AD and BB_1 meet in point M, segments BD and AA_1 meet in point N. Prove that the circumcircles of triangles B_1DM and A_1DN touch.

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2 In triangle $ABC, \angle B = 2\angle C$. Points P and Q on the medial perpendicular to CB are such that $\angle CAP = \angle PAQ = \angle QAB = \frac{\angle A}{3}$. Prove that Q is the circumcenter of triangle CPB.

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3 Restore the isosceles triangle ABC ($AB = AC$) if the common points I, M, H of bisectors, medians and altitudes respectively are given.

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4 Quadrilateral $ABCD$ is inscribed into a circle with center O. The bisectors of its angles form a cyclic quadrilateral with circumcenter I, and its external bisectors form a cyclic quadrilateral with circumcenter J. Prove that O is the midpoint of IJ.

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5 It is possible to compose a triangle from the altitudes of a given triangle. Can we conclude that it is possible to compose a triangle from its bisectors?

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6 In triangle ABC AA_0 and BB_0 are medians, AA_1 and BB_1 are altitudes. The circumcircles of triangles CA_0B_0 and CA_1B_1 meet again in point M_c. Points M_a, M_b are defined similarly. Prove that points M_a, M_b, M_c are collinear and lines AM_a, BM_b, CM_c are parallel.

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7 Circles ω and Ω are inscribed into the same angle. Line ℓ meets the sides of angles, ω and Ω in points A and F, B and C, D and E respectively (the order of points on the line is A, B, C, D, E, F). It is known that$BC = DE$. Prove that $AB = EF$.

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8 A convex n-gon P, where $n > 3$, is dissected into equal triangles by diagonals non-intersecting inside it. Which values of n are possible, if P is circumscribed?

– Grade 10

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1 In triangle ABC the midpoints of sides AC, BC, vertex C and the centroid lie on the same circle. Prove that this circle touches the circle passing through A, B and the orthocenter of triangle ABC.

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2 Quadrilateral $ABCD$ is circumscribed. Its incircle touches sides AB, BC, CD, DA in points K, L, M, N respectively. Points A', B', C', D' are the midpoints of segments LM, MN, NK, KL. Prove that the quadrilateral formed by lines AA', BB', CC', DD' is cyclic.

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3 Given two tetrahedrons $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Consider six pairs of edges A_iA_j and B_kB_l, where (i, j, k, l) is a transposition of numbers $(1, 2, 3, 4)$ (for example A_1A_2 and B_3B_4). It is known that for all but one such pairs the edges are perpendicular. Prove that the edges in the remaining pair also are perpendicular.

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4 Point D lies on the side AB of triangle ABC. The circle inscribed in angle ADC touches internally the circumcircle of triangle ACD. Another circle inscribed in angle BDC touches internally the circumcircle of triangle BCD. These two circles touch segment CD in the same point X. Prove that the perpendicular from X to AB passes through the incenter of triangle ABC

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5 The touching point of the excircle with the side of a triangle and the base of the altitude to this side are symmetric wrt the base of the corresponding bisector. Prove that this side is equal to one third of the perimeter.

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6 Prove that for any nonisosceles triangle $l_1^2 > \sqrt{3}S > l_2^2$, where l_1, l_2 are the greatest and the smallest bisectors of the triangle and S is its area.

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7 Point O is the circumcenter of acute-angled triangle ABC, points A_1, B_1, C_1 are the bases of its altitudes. Points A', B', C' lying on lines OA_1, OB_1, OC_1 respectively are such that quadrilaterals $AOBC', BOCA', COAB'$ are cyclic. Prove that the circumcircles of triangles AA_1A', BB_1B', CC_1C' have a common point.

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8 Given a sheet of tin 6×6. It is allowed to bend it and to cut it but in such a way that it doesnt fall to pieces. How to make a cube with edge 2, divided by partitions into unit cubes?
