Sharygin Geometry Olympiad 2012

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– **First (Correspondence) Round**

1 In triangle ABC point M is the midpoint of side AB, and point D is the foot of altitude CD. Prove that $\angle A = 2\angle B$ if and only if $AC = 2MD$.

2 A cyclic n-gon is divided by non-intersecting (inside the n-gon) diagonals to $n-2$ triangles. Each of these triangles is similar to at least one of the remaining ones. For what n this is possible?

3 A circle with center I touches sides AB, BC, CA of triangle ABC in points C_1, A_1, B_1. Lines AI, CI, B_1I meet A_1C_1 in points X, Y, Z respectively. Prove that $\angle YB_1Z = \angle XB_1Z$.

4 Given triangle ABC. Point M is the midpoint of side BC, and point P is the projection of B to the perpendicular bisector of segment AC. Line PM meets AB in point Q. Prove that triangle QPB is isosceles.

5 On side AC of triangle ABC an arbitrary point is selected D. The tangent in D to the circumcircle of triangle BDC meets AB in point C_1; point A_1 is defined similarly. Prove that $A_1C_1 \parallel AC$.

6 Point C_1 of hypothenuse AC of a right-angled triangle ABC is such that $BC = CC_1$. Point C_2 on cathetus AB is such that $AC_2 = AC_1$; point A_2 is defined similarly. Find angle AMC, where M is the midpoint of A_2C_2.

7 In a non-isosceles triangle ABC the bisectors of angles A and B are inversely proportional to the respective sidelengths. Find angle C.

8 Let BM be the median of right-angled triangle $ABC (\angle B = 90^\circ)$. The incircle of triangle ABM touches sides AB, AM in points A_1, A_2; points C_1, C_2 are defined similarly. Prove that lines A_1A_2 and C_1C_2 meet on the bisector of angle ABC.

9 In triangle ABC, given lines l_b and l_c containing the bisectors of angles B and C, and the foot L_1 of the bisector of angle A. Restore triangle ABC.

10 In a convex quadrilateral all sidelengths and all angles are pairwise different.
a) Can the greatest angle be adjacent to the greatest side and at the same time the smallest angle be adjacent to the smallest side?
b) Can the greatest angle be non-adjacent to the smallest side and at the same time the smallest angle be non-adjacent to the greatest side?

11 Given triangle ABC and point P. Points A', B', C' are the projections of P to BC, CA, AB. A line passing through P and parallel to AB meets the circumcircle of triangle $PA'B'$ for the second time in point C_1. Points A_1, B_1 are defined similarly. Prove that
a) lines AA_1, BB_1, CC_1 concur;
b) triangles ABC and $A_1B_1C_1$ are similar.

12 Let O be the circumcenter of an acute-angled triangle ABC. A line passing through O and parallel to BC meets AB and AC in points P and Q respectively. The sum of distances from O to AB and AC is equal to OA. Prove that $PB + QC = PQ$.

13 Points A, B are given. Find the locus of points C such that C, the midpoints of AC, BC and the centroid of triangle ABC are concyclic.

14 In a convex quadrilateral $ABCD$ suppose $AC \cap BD = O$ and M is the midpoint of BC. Let $MO \cap AD = E$. Prove that $\frac{AE}{ED} = \frac{S_{\triangle ABO}}{S_{\triangle CDO}}$.

15 Given triangle ABC. Consider lines l with the next property: the reflections of l in the sidelines of the triangle concur. Prove that all these lines have a common point.

16 Given right-angled triangle ABC with hypothenuse AB. Let M be the midpoint of AB and O be the center of circumcircle ω of triangle CMB. Line AC meets ω for the second time in point K. Segment KO meets the circumcircle of triangle ABC in point L. Prove that segments AL and KM meet on the circumcircle of triangle ACM.

17 A square $ABCD$ is inscribed into a circle. Point M lies on arc BC, AM meets BD in point P, DM meets AC in point Q. Prove that the area of quadrilateral $APQD$ is equal to the half of the area of the square.

18 A triangle and two points inside it are marked. It is known that one of the triangle's angles is equal to 58°, one of two remaining angles is equal to 59°, one of two given points is the incenter of the triangle and the second one is its circumcenter. Using only the ruler without partitions determine where is each of the angles and where is each of the centers.

19 Two circles with radii 1 meet in points X, Y, and the distance between these points also is equal to 1. Point C lies on the first circle, and lines CA, CB are tangents to the second one. These tangents meet the first circle for the second time in points B', A'. Lines AA' and BB' meet in point Z. Find angle XZY.

20 Point D lies on side AB of triangle ABC. Let ω_1 and Ω_1, ω_2 and Ω_2 be the incircles and the excircles (touching segment AB) of triangles ACD and BCD. Prove that the common external tangents to ω_1 and ω_2, Ω_1 and Ω_2 meet on AB.

21 Two perpendicular lines pass through the orthocenter of an acute-angled triangle. The sidelines of the triangle cut on each of these lines two segments: one lying inside the triangle and another one lying outside it. Prove that the product of two internal segments is equal to the product of two external segments.

Nikolai Beluhov and Emil Kolev

22 A circle ω with center I is inscribed into a segment of the disk, formed by an arc and a chord AB. Point M is the midpoint of this arc AB, and point N is the midpoint of the complementary arc. The tangents from N touch ω in points C and D. The opposite sidelines AC and BD of quadrilateral $ABCD$ meet in point X, and the diagonals of $ABCD$ meet in point Y. Prove that points X, Y, I and M are collinear.

23 An arbitrary point is selected on each of twelve diagonals of the faces of a cube.The centroid of these twelve points is determined. Find the locus of all these centroids.

24 Given are n $(n > 2)$ points on the plane such that no three of them are collinear. In how many ways this set of points can be divided into two non-empty subsets with non-intersecting convex envelops?

– **Final Round**

– grade 8

1 Let M be the midpoint of the base AC of an acute-angled isosceles triangle ABC. Let N be the reflection of M in BC. The line parallel to AC and passing through N meets AB at point K. Determine the value of $\angle AKC$.

(A.Blinkov)

2 In a triangle ABC the bisectors BB' and CC' are drawn. After that, the whole picture except the points A, B', and C' is erased. Restore the triangle using a compass and a ruler.

(A.Karlyuchenko)

3 A paper square was bent by a line in such way that one vertex came to a side not containing this vertex. Three circles are inscribed into three obtained triangles (see Figure). Prove that one of their radii is equal to the sum of the two remaining ones.

(L.Steingarts)

4 Let ABC be an isosceles triangle with $\angle B = 120^o$. Points P and Q are chosen on the prolongations of segments AB and CB beyond point B so that the rays AQ and CP intersect and are perpendicular to each other. Prove that $\angle PQB = 2\angle PCQ$.

(A.Akopyan, D.Shvetsov)

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5 Do there exist a convex quadrilateral and a point P inside it such that the sum of distances from P to the vertices of the quadrilateral is greater than its perimeter?

(A.Akopyan)

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6 Let ω be the circumcircle of triangle ABC. A point B_1 is chosen on the prolongation of side AB beyond point B so that $AB_1 = AC$. The angle bisector of $\angle BAC$ meets ω again at point W. Prove that the orthocenter of triangle AWB_1 lies on ω .

(A.Tumanyan)

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7 The altitudes AA_1 and CC_1 of an acute-angled triangle ABC meet at point H. Point Q is the reflection of the midpoint of AC in line AA_1, point P is the midpoint of segment A_1C_1. Prove that $\angle QPH = 90^o$.

(D.Shvetsov)

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8 A square is divided into several (greater than one) convex polygons with mutually different numbers of sides. Prove that one of these polygons is a triangle.

(A.Zaslavsky)

– Grade 9

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1 The altitudes AA_1 and BB_1 of an acute-angled triangle ABC meet at point O. Let A_1A_2 and B_1B_2 be the altitudes of triangles OBA_1 and OAB_1 respectively. Prove that A_2B_2 is parallel to AB.

(L.Steingarts)

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2 Three parallel lines passing through the vertices A, B, and C of triangle ABC meet its circumcircle again at points A_1, B_1, and C_1 respectively. Points A_2, B_2, and C_2 are the reflections of points A_1, B_1, and C_1 in BC, CA, and AB respectively. Prove that the lines AA_2, BB_2, CC_2 are concurrent.

(D.Shvetsov, A.Zaslavsky)

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3 In triangle ABC, the bisector CL was drawn. The incircles of triangles CAL and CBL touch AB at points M and N respectively. Points M and N are marked on the picture, and then the whole picture except the points A, L, M, and N is erased. Restore the triangle using a compass and a ruler.

(V.Protasov)

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4 Determine all integer $n > 3$ for which a regular n-gon can be divided into equal triangles by several (possibly intersecting) diagonals.

(B.Frenkin)

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5 Let ABC be an isosceles right-angled triangle. Point D is chosen on the prolongation of the hypothenuse AB beyond point A so that $AB = 2AD$. Points M and N on side AC satisfy the relation $AM = NC$. Point K is chosen on the prolongation of CB beyond point B so that $CN = BK$. Determine the angle between lines NK and DM.

(M.Kungozhin)

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6 Let ABC be an isosceles triangle with $BC = a$ and $AB = AC = b$. Segment AC is the base of an isosceles triangle ADC with $AD = DC = a$ such that points D and B share the opposite sides of AC. Let CM and CN be the bisectors in triangles ABC and ADC respectively. Determine the circumradius of triangle CMN.

(M.Rozhkova)

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7 A convex pentagon P is divided by all its diagonals into ten triangles and one smaller pentagon P'. Let N be the sum of areas of five triangles adjacent to the sides of P decreased by the area of P'. The same operations are performed with the pentagon P', let N' be the similar difference calculated for this pentagon. Prove that $N > N'$.

(A.Belov)

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8 Let AH be an altitude of an acute-angled triangle ABC. Points K and L are the projections of H onto sides AB and AC. The circumcircle of ABC meets line KL at points P and Q, and meets line AH at points A and T. Prove that H is the incenter of triangle PQT.

(M.Plotnikov)

– Grade 10

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1 Determine all integer n such that a surface of an $n \times n \times n$ grid cube can be pasted in one layer by paper 1×2 rectangles so that each rectangle has exactly five neighbors (by a line segment).

(A.Shapovalov)

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2 We say that a point inside a triangle is good if the lengths of the cevians passing through this point are inversely proportional to the respective side lengths. Find all the triangles for which the number of good points is maximal.

(A.Zaslavsky, B.Frenkin)

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3 Let M and I be the centroid and the incenter of a scalene triangle ABC, and let r be its inradius. Prove that $MI = r/3$ if and only if MI is perpendicular to one of the sides of the triangle.

(A.Karlyuchenko)

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4 Consider a square. Find the locus of midpoints of the hypothenuses of rightangled triangles with the vertices lying on three different sides of the square and not coinciding with its vertices.

(B.Frenkin)

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5 A quadrilateral $ABCD$ with perpendicular diagonals is inscribed into a circle ω. Two arcs α and β with diameters AB and CD lie outside ω. Consider two crescents formed by the circle ω and the arcs α and β (see Figure). Prove that the maximal radii of the circles inscribed into these crescents are equal.

(F.Nilov)

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6 Consider a tetrahedron $ABCD$. A point X is chosen outside the tetrahedron so that segment XD intersects face ABC in its interior point. Let A', B' , and C' be the projections of D onto the planes XBC, XCA, and XAB respectively. Prove that $A'B' + B'C' + C'A' \leq DA + DB + DC$.

(V.Yassinsky)

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7 Consider a triangle ABC. The tangent line to its circumcircle at point C meets line AB at point D. The tangent lines to the circumcircle of triangle ACD at points A and C meet at point K. Prove that line DK bisects segment BC.

(F.Ivlev)

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8 A point M lies on the side BC of square $ABCD$. Let X, Y , and Z be the incenters of triangles ABM, CMD, and AMD respectively. Let H_x, H_y, and H_z be the orthocenters of triangles AXB, CYD, and AZD. Prove that H_x, H_y, and H_z are collinear.
