Sharygin Geometry Olympiad 2013

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–	First Round
1	Let ABC be an isosceles triangle with $AB = BC$. Point E lies on the side AB, and ED is the perpendicular from E to BC. It is known that $AE = DE$. Find $\angle DAC$.
2	Let ABC be an isosceles triangle ($AC = BC$) with $\angle C = 20^\circ$. The bisectors of angles A and B meet the opposite sides at points A_1 and B_1 respectively. Prove that the triangle A_1OB_1 (where O is the circumcenter of ABC) is regular.
3	Let ABC be a right-angled triangle ($\angle B = 90^\circ$). The excircle inscribed into the angle A touches the extensions of the sides AB, AC at points A_1, A_2 respectively; points C_1, C_2 are defined similarly. Prove that the perpendiculars from A, B, C to C_1C_2, A_1C_1, A_1A_2 respectively, concur.
4	Let ABC be a nonisosceles triangle. Point O is its circumcenter, and point K is the center of the circumcircle w of triangle BCO. The altitude of ABC from A meets w at a point P. The line PK intersects the circumcircle of ABC at points E and F. Prove that one of the segments EP and FP is equal to the segment PA.
5	Four segments drawn from a given point inside a convex quadrilateral to its vertices, split the quadrilateral into four equal triangles. Can we assert that this quadrilateral is a rhombus?
6	Diagonals AC and BD of a trapezoid $ABCD$ meet at P. The circumcircles of triangles ABP and CDP intersect the line AD for the second time at points X and Y respectively. Let M be the midpoint of segment XY. Prove that $BM = CM$.
7	Let BD be a bisector of triangle ABC. Points I_a, I_c are the incenters of triangles ABD, CBD respectively. The line I_aI_c meets AC in point Q. Prove that $\angle DBQ = 90^\circ$.
8	Let X be an arbitrary point inside the circumcircle of a triangle ABC. The lines BX and CX meet the circumcircle in points K and L respectively. The line LK intersects BA and AC at points E and F respectively. Find the locus of points X such that the circumcircles of triangles AFK and AEL touch.
9	Let T_1 and T_2 be the points of tangency of the excircles of a triangle ABC with its sides BC and AC respectively. It is known that the reflection of the incenter of ABC across the midpoint of AB lies on the circumcircle of triangle CT_1T_2. Find $\angle BCA$.

10 The incircle of triangle ABC touches the side AB at point C'; the incircle of triangle ACC' touches the sides AB and AC at points C_1, B_1; the incircle of triangle BCC' touches the sides AB and BC at points C_2, A_2. Prove that the lines B_1C_1, A_2C_2, and CC' concur.

11 a) Let $ABCD$ be a convex quadrilateral and $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triangles ABC, BCD, CDA, DAB. Can the inequality $r_4 > 2r_3$ hold?

b) The diagonals of a convex quadrilateral $ABCD$ meet in point E. Let $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triangles ABE, BCE, CDE, DAE. Can the inequality $r_2 > 2r_1$ hold?

12 On each side of triangle ABC, two distinct points are marked. It is known that these points are the feet of the altitudes and of the bisectors.

a) Using only a ruler determine which points are the feet of the altitudes and which points are the feet of the bisectors.

b) Solve p.a) drawing only three lines.

13 Let A_1 and C_1 be the tangency points of the incircle of triangle ABC with BC and AB respectively, A' and C' be the tangency points of the excircle inscribed into the angle B with the extensions of BC and AB respectively. Prove that the orthocenter H of triangle ABC lies on A_1C_1 if and only if the lines $A'C_1$ and BA are orthogonal.

14 Let M, N be the midpoints of diagonals AC, BD of a right-angled trapezoid $ABCD$ ($\measuredangle A = \measuredangle D = 90^\circ$).
The circumcircles of triangles ABN, CDM meet the line BC in the points Q, R.
Prove that the distances from Q, R to the midpoint of MN are equal.

15 (a) Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are inscribed into triangle ABC so that $C_1A_1 \perp BC$, $A_1B_1 \perp CA$, $B_1C_1 \perp AB$, $B_2A_2 \perp BC$, $C_2B_2 \perp CA$, $A_2C_2 \perp AB$. Prove that these triangles are equal.

(b) Points A_1, B_1, C_1, A_2, B_2, C_2 lie inside a triangle ABC so that A_1 is on segment AB_1, B_1 is on segment BC_1, C_1 is on segment CA_1, A_2 is on segment AC_2, B_2 is on segment BA_2, C_2 is on segment CB_2, and the angles BAA_1, CBB_2, ACC_1, CAA_2, ABB_2, BCC_2 are equal. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equal.

16 The incircle of triangle ABC touches BC, CA, AB at points A_1, B_1, C_1, respectively. The perpendicular from the incenter I to the median from vertex C meets the line A_1B_1 in point K. Prove that CK is parallel to AB.

17 An acute angle between the diagonals of a cyclic quadrilateral is equal to ϕ. Prove that an acute angle between the diagonals of any other quadrilateral having the same sidelengths is smaller than ϕ.

18 Let AD be a bisector of triangle ABC. Points M and N are projections of B and C respectively to AD. The circle with diameter MN intersects BC at points X and Y. Prove that $\angle BAX = \angle CAY$.

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19 a) The incircle of a triangle ABC touches AC and AB at points B_0 and C_0 respectively. The bisectors of angles B and C meet the perpendicular bisector to the bisector AL in points Q and P respectively. Prove that the lines PC_0, QB_0 and BC concur.

b) Let AL be the bisector of a triangle ABC. Points O_1 and O_2 are the circumcenters of triangles ABL and ACL respectively. Points B_1 and C_1 are the projections of C and B to the bisectors of angles B and C respectively. Prove that the lines $O_1C_1, O_2B_1,$ and BC concur.

c) Prove that the two points obtained in pp. a) and b) coincide.

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20 Let C_1 be an arbitrary point on the side AB of triangle ABC. Points A_1 and B_1 on the rays BC and AC are such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. The lines AA_1 and BB_1 meet in point C_2. Prove that all the lines C_1C_2 have a common point.

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21 Chords BC and DE of circle ω meet at point A. The line through D parallel to BC meets ω again at F, and FA meets ω again at T. Let $M = ET \cap BC$ and let N be the reflection of A over M. Show that (DEN) passes through the midpoint of BC.

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22 The common perpendiculars to the opposite sidelines of a nonplanar quadrilateral are mutually orthogonal. Prove that they intersect.

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23 Two convex polytopes A and B do not intersect. The polytope A has exactly 2012 planes of symmetry. What is the maximal number of symmetry planes of the union of A and B, if B has a) 2012, b) 2013 symmetry planes?

c) What is the answer to the question of p.b), if the symmetry planes are replaced by the symmetry axes?

– Final Round

– Grade level 8

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1 Let $ABCDE$ be a pentagon with right angles at vertices B and E and such that $AB = AE$ and $BC = CD = DE$. The diagonals BD and CE meet at point F. Prove that $FA = AB$.

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2 Two circles with centers O_1 and O_2 meet at points A and B. The bisector of angle O_1AO_2 meets the circles for the second time at points $C$and D. Prove that the distances from the circumcenter of triangle CBD to O_1 and to O_2 are equal.

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3 Each vertex of a convex polygon is projected to all nonadjacent sidelines. Can it happen that each of these projections lies outside the corresponding side?

4 The diagonals of a convex quadrilateral $ABCD$ meet at point L. The orthocenter H of the triangle LAB and the circumcenters O_1, O_2, and O_3 of the triangles LBC, LCD, and LDA were marked. Then the whole configuration except for points H, O_1, O_2, and O_3 was erased. Restore it using a compass and a ruler.

5 The altitude AA', the median BB', and the angle bisector CC' of a triangle ABC are concurrent at point K. Given that $A'K = B'K$, prove that $C'K = A'K$.

6 Dear Mathlinkers,

1. A, B the end points of an arch circle
2. (O) a circle tangent to AB intersecting the arch in question
3. T the point of contact of (O) and AB
4. C, D the points of intersection of (O) with the arch in the order A, D, C, B
5. E, F the points of intersection of AC and DT, BD and CT.

Prove : EF is parallel to AB.

Sincerely
Jean-Louis

7 In the plane, four points are marked. It is known that these points are the centers of four circles, three of which are pairwise externally tangent, and all these three are internally tangent to the fourth one. It turns out, however, that it is impossible to determine which of the marked points is the center of the fourth (the largest) circle. Prove that these four points are the vertices of a rectangle.

8 Let P be an arbitrary point on the arc AC of the circumcircle of a fixed triangle ABC, not containing B. The bisector of angle APB meets the bisector of angle BAC at point P_a the bisector of angle CPB meets the bisector of angle BCA at point P_c. Prove that for all points P, the circumcenters of triangles PP_aP_c are collinear.

by I. Dmitriev

– Grade level 9

1 All angles of a cyclic pentagon $ABCDE$ are obtuse. The sidelines AB and CD meet at point E_1, the sidelines BC and DE meet at point A_1. The tangent at B to the circumcircle of the triangle BE_1C meets the circumcircle ω of the pentagon for the second time at point B_1. The tangent at D to the circumcircle of the triangle DA_1C meets ω for the second time at point D_1. Prove that $B_1D_1 // AE$

2 Two circles ω_1 and ω_2 with centers O_1 and O_2 meet at points A and B. Points C and D on ω_1 and ω_2, respectively, lie on the opposite sides of the line AB and are equidistant from this line.

Prove that C and D are equidistant from the midpoint of O_1O_2.

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3 Each sidelength of a convex quadrilateral $ABCD$ is not less than 1 and not greater than 2. The diagonals of this quadrilateral meet at point O. Prove that $S_{AOB} + S_{COD} \le 2(S_{AOD} + S_{BOC})$.

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4 A point F inside a triangle ABC is chosen so that $\angle AFB = \angle BFC = \angle CFA$. The line passing through F and perpendicular to BC meets the median from A at point A_1. Points B_1 and C_1 are defined similarly. Prove that the points A_1, B_1, and C_1 are three vertices of some regular hexagon, and that the three remaining vertices of that hexagon lie on the sidelines of ABC.

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5 Points E and F lie on the sides AB and AC of a triangle ABC. Lines EF and BC meet at point S. Let M and N be the midpoints of BC and EF, respectively. The line passing through A and parallel to MN meets BC at point K. Prove that $\frac{BK}{CK} = \frac{FS}{ES}$.
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6 A line ℓ passes through the vertex B of a regular triangle ABC. A circle ω_a centered at I_a is tangent to BC at point A_1, and is also tangent to the lines ℓ and AC. A circle ω_c centered at I_c is tangent to BA at point C_1, and is also tangent to the lines ℓ and AC. Prove that the orthocenter of triangle A_1BC_1 lies on the line I_aI_c.

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7 Two fixed circles ω_1 and ω_2 pass through point O. A circle of an arbitrary radius R centered at O meets ω_1 at points A and B, and meets ω_2 at points C and D. Let X be the common point of lines AC and BD. Prove that all the points X are collinear as R changes.

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8 Three cyclists ride along a circular road with radius 1 km counterclockwise. Their velocities are constant and different. Does there necessarily exist (in a sufficiently long time) a moment when all the three distances between cyclists are greater than 1 km?

by V. Protasov

– Grade level 10

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1 A circle k passes through the vertices B, C of a scalene triangle ABC. k meets the extensions of AB, AC beyond B, C at P, Q respectively. Let A_1 is the foot the altitude drop from A to BC. Suppose $A_1P = A_1Q$. Prove that $\widehat{PA_1Q} = 2\widehat{BAC}$.

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2 Let $ABCD$ is a tangential quadrilateral such that $AB = CD > BC$. AC meets BD at L. Prove that $\widehat{ALB}$ is acute.

According to the jury, they want to propose a more generalized problem is to prove $(AB - CD)^2 < (AD - BC)^2$, but this problem has appeared some time ago

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3 Let X be a point inside triangle ABC such that $XA.BC = XB.AC = XC.AC$. Let I_1, I_2, I_3 be the incenters of XBC, XCA, XAB. Prove that AI_1, BI_2, CI_3 are concurrent.

Of course, the most natural way to solve this is the Ceva sin theorem, but there is an another approach that may surprise you;), try not to use the Ceva theorem :))

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4 Given a square cardboard of area $\frac{1}{4}$, and a paper triangle of area $\frac{1}{2}$ such that the square of its sidelength is a positive integer. Prove that the triangle can be folded in some ways such that the squace can be placed inside the folded figure so that both of its faces are completely covered with paper.

Proposed by N.Beluhov, Bulgaria

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5 Let ABCD is a cyclic quadrilateral inscribed in (O). E, F are the midpoints of arcs AB and CD not containing the other vertices of the quadrilateral. The line passing through E, F and parallel to the diagonals of $ABCD$ meet at E, F, K, L. Prove that KL passes through O.

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6 The altitudes AA_1, BB_1, CC_1 of an acute triangle ABC concur at H. The perpendicular lines from H to B_1C_1, A_1C_1 meet rays CA, CB at P, Q respectively. Prove that the line from C perpendicular to A_1B_1 passes through the midpoint of PQ.

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7 Given five fixed points in the space. It is known that these points are centers of five spheres, four of which are pairwise externally tangent, and all these point are internally tangent to the fifth one. It turns out that it is impossible to determine which of the marked points is the center of the largest sphere. Find the ratio of the greatest and the smallest radii of the spheres.

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8 Two fixed circles are given on the plane, one of them lies inside the other one. From a point C moving arbitrarily on the external circle, draw two chords CA, CB of the larger circle such that they tangent to the smalaler one. Find the locus of the incenter of triangle ABC.
