National Mathematical Olympiad 2010

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1 Let CD be a chord of a circle Γ_1 and AB a diameter of Γ_1 perpendicular to CD at N with $AN > NB$. A circle Γ_2 centered at C with radius CN intersects Γ_1 at points P and Q. The line PQ intersects CD at M and AC at K; and the extension of NK meets Γ_2 at L. Prove that PQ is perpendicular to AL

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2 Let $(a_n), (b_n)$, $n = 1, 2, ...$ be two sequences of integers defined by $a_1 = 1, b_1 = 0$ and for $n \geq 1$

$a_{n+1} = 7a_n + 12b_n + 6$ $b_{n+1} = 4a_n + 7b_n + 3$

Prove that a_n^2 is the difference of two consecutive cubes.

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3 Suppose that $a_1, ..., a_{15}$ are prime numbers forming an arithmetic progression with common difference $d > 0$ if $a_1 > 15$ show that $d > 30000$

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4 Let n be a positive integer. Find the smallest positive integer k with the property that for any colouring nof the squares of a $2n$ by k chessboard with n colours, there are 2 columns and 2 rows such that the 4 squares in their intersections have the same colour.

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5 A prime number p and integers x, y, z with $0 < x < y < z < p$ are given. Show that if the numbers x^3, y^3, z^3 give the same remainder when divided by p, then $x^2 + y^2 + z^2$ is divisible by $x + y + z$.
