National Mathematical Olympiad 2011

www.artofproblemsolving.com/community/c3687

by oneplusone

1 In the acute-angled non-isosceles triangle ABC, O is its circumcenter, H is its orthocenter and $AB > AC$. Let Q be a point on AC such that the extension of HQ meets the extension of BC at the point P. Suppose $BD = DP$, where D is the foot of the perpendicular from A onto BC. Prove that $\angle ODQ = 90^\circ$.

---

2 If 46 squares are colored red in a 9×9 board, show that there is a 2×2 block on the board in which at least 3 of the squares are colored red.

---

3 Let $x, y, z > 0$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{1}{xyz}$. Show that

$$\frac{2x}{\sqrt{1+x^2}} + \frac{2y}{\sqrt{1+y^2}} + \frac{2z}{\sqrt{1+z^2}} < 3.$$

---

4 Find all polynomials $P(x)$ with real coefficients such that

$$P(a) \in \mathbb{Z} \text{ implies that } a \in \mathbb{Z}.$$

---

5 Find all pairs of positive integers (m, n) such that

$$m + n - \frac{3mn}{m+n} = \frac{2011}{3}.$$
