National Mathematical Olympiad 2012

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by 61plus

1 The incircle with centre I of the triangle ABC touches the sides BC, CA and AB at D, E, F respectively. The line ID intersects the segment EF at K. Proof that A, K, M collinear, where M is the midpoint of BC.

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2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that $(x + y)(f(x) - f(y)) = (x - y)f(x + y)$ for all x, y that belongs to $\mathbb{R}$.

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3 For each $i = 1, 2, ..N$, let a_i, b_i, c_i be integers such that at least one of them is odd. Show that one can find integers x, y, z such that $xa_i + yb_i + zc_i$ is odd for at least $\frac{4}{7}N$ different values of i.

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4 Let p be an odd prime. Prove that

$$1^{p-2} + 2^{p-2} + \cdots + \left(\frac{p-1}{2}\right)^{p-2} \equiv \frac{2 - 2^p}{p} \pmod{p}.$$

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5 There are 2012 distinct points in the plane, each of which is to be coloured using one of n colours, so that the numbers of points of each colour are distinct. A set of n points is said to be *multi-coloured* if their colours are distinct. Determine n that maximizes the number of multi-coloured sets.
