Singapore Team Selection Test 2004

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by Valiowk

Day 1

1 Let D be a point in the interior of $\triangle ABC$ such that $AB = ab$, $AC = ac$, $AD = ad$, $BC = bc$, $BD = bd$ and $CD = cd$. Prove that $\angle ABD + \angle ACD = \frac{\pi}{3}$.

2 Let $0 < a, b, c < 1$ with $ab + bc + ca = 1$. Prove that

$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} \geq \frac{3\sqrt{3}}{2}.$$

Determine when equality holds.

3 Let $p \geq 5$ be a prime number. Prove that there exist at least 2 distinct primes q_1, q_2 satisfying $1 < q_i < p - 1$ and $q_i^{p-1} \not\equiv 1$ (mod p^2), for $i = 1, 2$.

Day 2

1 Let $x_0, x_1, x_2, \ldots$ be the sequence defined by $x_i = 2^i$ if $0 \leq i \leq 2003$ $x_i = \sum_{j=1}^{2004} x_{i-j}$ if $i \geq 2004$

Find the greatest k for which the sequence contains k consecutive terms divisible by 2004.

2 Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

Proposed by Hojoo Lee, Korea

3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f\left(\frac{x+y}{x-y}\right) = \frac{f(x) + f(y)}{f(x) - f(y)}$$

for all $x \neq y$.