Singapore Team Selection Test 2007

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Day 1

1 Find all pairs of nonnegative integers (x, y) satisfying $(14y)^x + y^{x+y} = 2007$.

2 Let $ABCD$ be a convex quadrilateral inscribed in a circle with M and N the midpoints of the diagonals AC and BD respectively. Suppose that AC bisects $\angle BMD$. Prove that BD bisects $\angle ANC$.

3 Let $a_1, a_2, \dots, a_8$ be 8 distinct points on the circumference of a circle such that no three chords, each joining a pair of the points, are concurrent. Every 4 of the 8 points form a quadrilateral which is called a *quad*. If two chords, each joining a pair of the 8 points, intersect, the point of intersection is called a *bullet*. Suppose some of the bullets are coloured red. For each pair (ij), with $1 \le i < j \le 8$, let $r(i, j)$ be the number of quads, each containing a_i, a_j as vertices, whose diagonals intersect at a red bullet. Determine the smallest positive integer n such that it is possible to colour n of the bullets red so that $r(i, j)$ is a constant for all pairs (i, j).

Day 2

1 Two circles (O_1) and (O_2) touch externally at the point C and internally at the points A and B respectively with another circle (O). Suppose that the common tangent of (O_1) and (O_2) at C meets (O) at P such that $PA = PB$. Prove that PO is perpendicular to AB.

2 Prove the inequality

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \le \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{i<j} a_i a_j$$

for all positive real numbers $a_1, a_2, \dots, a_n$.

3 Let A, B, C be 3 points on the plane with integral coordinates. Prove that there exists a point P with integral coordinates distinct from A, B and C such that the interiors of the segments PA, PB and PC do not contain points with integral coordinates.