Singapore Team Selection Test 2008

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Day 1

1 In triangle ABC, D is a point on AB and E is a point on AC such that BE and CD are bisectors of $\angle B$ and $\angle C$ respectively. Let Q, M and N be the feet of perpendiculars from the midpoint P of DE onto BC, AB and AC, respectively. Prove that $PQ = PM + PN$.

2 Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 x_2 \cdots x_n = 1$. Prove that

$$\sum_{i=1}^{n} \frac{1}{n-1+x_i} \leq 1.$$

3 Find all odd primes p, if any, such that p divides $\sum_{n=1}^{103} n^{p-1}$

Day 2

1 Let (O) be a circle, and let ABP be a line segment such that A, B lie on (O) and P is a point outside (O). Let C be a point on (O) such that PC is tangent to (O) and let D be the point on (O) such that CD is a diameter of (O) and intersects AB inside (O). Suppose that the lines DB and OP intersect at E. Prove that AC is perpendicular to CE.

2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $(x+y)(f(x)-f(y)) = (x-y)f(x+y)$ for all $x, y \in \mathbb{R}$

3 Fifty teams participate in a round robin competition over 50 days. Moreover, all the teams (at least two) that show up in any day must play against each other. Prove that on every pair of consecutive days, there is a team that has to play on those two days.