Mikls Schweitzer 1949

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1 Let an infinite sequence of measurable sets be given on the interval $(0,1)$ the measures of which are $\geq \alpha > 0$. Show that there exists a point of $(0,1)$ which belongs to infinitely many terms of the sequence.

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2 Compute $\lim_{n\to\infty} \int_0^{\pi} \frac{\sin x}{1+\cos^2 nx} dx$.

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3 Let p be an odd prime number and $a_1, a_2, ..., a_p$ and $b_1, b_2, ..., b_p$ two arbitrary permutations of the numbers $1, 2, ..., p$. Show that the least positive residues modulo p of the numbers $a_1b_1, a_2b_2, ..., a_pb_p$ never form a permutation of the numbers $1, 2, ..., p$.

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4 Let A and B be two disjoint sets in the interval $(0,1)$. Denoting by μ the Lebesgue measure on the real line, let $\mu(A) > 0$ and $\mu(B) > 0$. Let further n be a positive integer and $\lambda = \frac{1}{n}$. Show that there exists a subinterval (c,d) of $(0,1)$ for which $\mu(A \cap (c,d)) = \lambda\mu(A)$ and $\mu(B \cap (c,d)) = \lambda\mu(B)$. Show further that this is not true if λ is not of the form $\frac{1}{n}$.

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5 Let $f(x)$ be a polynomial of second degree the roots of which are contained in the interval $[-1,+1]$ and let there be a point $x_0 \in [-1.+1]$ such that $|f(x_0)| = 1$. Prove that for every $\alpha \in [0,1]$, there exists a $\zeta \in [-1,+1]$ such that $|f'(\zeta)| = \alpha$ and that this statement is not true if $\alpha > 1$.

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6 Let n and k be positive integers, $n \geq k$. Prove that the greatest common divisor of the numbers $\binom{n}{k}, \binom{n+1}{k}, \ldots, \binom{n+k}{k}$ is 1.

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7 Find the complex numbers z for which the series

$$1 + \frac{z}{2!} + \frac{z(z+1)}{3!} + \frac{z(z+1)(z+2)}{4!} + \cdots + \frac{z(z+1)\cdots(z+n)}{(n+2)!} + \cdots$$

converges and find its sum.

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8 The four sides of a skew quadrangle and the two segments joining the midpoints of the opposite sides are realized by rigid bars. The bars are linked by hinges. Prove that this apparatus is not rigid.

P.S: The 1949 Miklos Schweitzer competition had only 8 problems!
