Mikls Schweitzer 1950

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by Albanian Eagle

Day 1

1 Let $\{k_n\}_{n=1}^{\infty}$ be a sequence of real numbers having the properties $k_1 > 1$ and $k_1+k_2+\cdots+k_n < 2k_n$ for $n = 1, 2, ...$ Prove that there exists a number $q > 1$ such that $k_n > q^n$ for every positive integer n.

2 Consider three different planes and consider also one point on each of them. Give necessary and sufficient conditions for the existence of a quadratic which passes through the given points and whose tangent-plane at each of these points is the respective given plane.

3 Let E be a system of n^2+1 closed intervals of the real line. Show that E has either a subsystem consisting of $n+1$ elements which are monotonically ordered with respect to inclusion or a subsystem consisting of $n+1$ elements none of which contains another element of the subsystem.

4 Find the polynomials $f(x)$ having the following properties:

(i) $f(0) = 1$, $f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0$
(ii) $f(1) = f'(1) = f''(1) = \cdots = f^{(m)}(1) = 0$

5 Prove that for every positive integer k there exists a sequence of k consecutive positive integers none of which can be represented as the sum of two squares.

6 Prove the following identity for determinants: $|c_{ik}+a_i+b_k+1|_{i,k=1,...,n} + |c_{ik}|_{i,k=1,...,n} = |c_{ik}+a_i+b_k|_{i,k=1,...,n} + |c_{ik}+1|_{i,k=1,...,n}$

7 Let x be an arbitrary real number in $(0,1)$. For every positive integer k, let $f_k(x)$ be the number of points $mx \in [k, k+1)$ $m = 1, 2, ...$
Show that the sequence $\sqrt[n]{f_1(x)f_2(x)\cdots f_n(x)}$ is convergent and find its limit.

8 Let $A = (a_{ik})$ be an $n \times n$ matrix with nonnegative elements such that $\sum_{k=1}^{n} a_{ik} = 1$ for $i = 1, ..., n$.
Show that, for every eigenvalue λ of A, either $|\lambda| < 1$ or there exists a positive integer k such that $\lambda^k = 1$

9 Find the sum of the series $x + \frac{x^3}{1\cdot 3} + \frac{x^5}{1\cdot 3\cdot 5} + \cdots + \frac{x^{2n+1}}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n+1)} + \cdots$

10 Consider an arc of a planar curve such that the total curvature of the arc is less than π. Suppose, further, that the curvature and its derivative with respect to the arc length exist at every point of the arc and the latter nowhere equals zero. Let the osculating circles belonging to the endpoints of the arc and one of these points be given. Determine the possible positions of the other endpoint.

Day 2

1 Let $a > 0, d > 0$ and put $f(x) = \frac{1}{a} + \frac{x}{a(a+d)} + \cdots + \frac{x^n}{a(a+d)\cdots(a+nd)} + \cdots$
Give a closed form for $f(x)$.

2 Show that there exists a positive constant c with the following property: To every positive irrational α, there can be found infinitely many fractions $\frac{p}{q}$ with $(p,q) = 1$ satisfying $\left|\alpha - \frac{p}{q}\right| \le \frac{c}{q^2}$

3 For any system $x_1, x_2, ..., x_n$ of positive real numbers, let $f_1(x_1, x_2, ..., x_n) = x_1$,
and $f_\nu = \frac{x_1+2x_2+\cdots+\nu x_\nu}{\nu+(\nu-1)x_1+(\nu-2)x_2+\cdots+1\cdot x_{\nu-1}}$

for $\nu = 2, 3, ..., n$. Show that for any $\epsilon > 0$, a positive integer $n_0 < n_0(\epsilon)$ can be found such that for every $n > n_0$ there exists a system $x_1', x_2', ..., x_n'$ of positive real numbers
with $x_1' + x_2' + \cdots + x_n' = 1$ and $f_\nu(x_1', x_2', ..., x_n') \le \epsilon$ for $\nu = 1, 2, ..., n$.

4 Put $M = \begin{pmatrix} p & q & r \\ r & p & q \\ q & r & p \end{pmatrix}$

where $p, q, r > 0$ and $p + q + r = 1$. Prove that $\lim_{n\to\infty} M^n = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

5 Let $1 \le a_1 < a_2 < \cdots < a_m \le N$ be a sequence of integers such that the least common multiple of any two of its elements is not greater than N. Show that $m \le 2\left[\sqrt{N}\right]$, where $\left[\sqrt{N}\right]$ denotes the greatest integer $\le \sqrt{N}$

6 Consider an arc of a planar curve; let the radius of curvature at any point of the arc be a differentiable function of the arc length and its derivative be everywhere different from zero; moreover, let the total curvature be less than $\frac{\pi}{2}$. Let P_1, P_2, P_3, P_4, P_5 and P_6 be any points on this arc, subject to the only condition that the radius of curvature at P_k is greater than at P_j if $j < k$.
Prove that the radius of the circle passing through the points P_1, P_3 and P_5 is less than the radius of the circle through P_2, P_4 and P_6

7 Examine the behavior of the expression $\sum_{\nu=1}^{n-1} \frac{\log(n-\nu)}{\nu} - \log^2 n$

as $n \to \infty$

8 A coastal battery sights an enemy cruiser lying one kilometer off the coast and opens fire on it at the rate of one round per minute. After the first shot, the cruiser begins to move away at a speed of 60 kilometers an hour. Let the probability of a hit be $0.75x^{-2}$, where x denotes the distance (in kilometers) between the cruiser and the coast ($x \geq 1$), and suppose that the battery goes on firing till the cruiser either sinks or disappears. Further, let the probability of the cruiser sinking after n hits be $1-\frac{1}{4^n}$ ($n=0,1,...$). Show that the probability of the cruiser escaping is $\frac{2\sqrt{2}}{3\pi}$

9 Find the necessary and sufficient conditions for two conics that every tangent to one of them contains a real point of the other.