Mikls Schweitzer 1951

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by Albanian Eagle

1 Choose terms of the harmonic series so that the sum of the chosen terms be finite. Prove that the sequence of these terms is of density zero in the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$

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2 Denote by $\mathcal{H}$ a set of sequences $S = \{s_n\}_{n=1}^{\infty}$ of real numbers having the following properties:

(i) If $S = \{s_n\}_{n=1}^{\infty} \in \mathcal{H}$, then $S' = \{s_n\}_{n=2}^{\infty} \in \mathcal{H}$;

(ii) If $S = \{s_n\}_{n=1}^{\infty} \in \mathcal{H}$ and $T = \{t_n\}_{n=1}^{\infty}$, then
$S + T = \{s_n + t_n\}_{n=1}^{\infty} \in \mathcal{H}$ and $ST = \{s_n t_n\}_{n=1}^{\infty} \in \mathcal{H}$;

(iii) $\{-1, -1, \dots, -1, \dots\} \in \mathcal{H}$.

A real valued function $f(S)$ defined on $\mathcal{H}$ is called a quasi-limit of S if it has the following properties:

If $S = c, c, \dots, c, \dots$, then $f(S) = c$;
If $s_i \geq 0$, then $f(S) \geq 0$; $f(S + T) = f(S) + f(T)$; $f(ST) = f(S)f(T)$, $f(S') = f(S)$

Prove that for every S, the quasi-limit $f(S)$ is an accumulation point of S.

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3 Consider the iterated sequence
(1) $x_0, x_1 = f(x_0), \dots, x_{n+1} = f(x_n), \dots$,
where $f(x) = 4x - x^2$. Determine the points x_0 of $[0, 1]$ for which (1) converges and find the limit of (1).

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4 Prove that the infinite series $1 - \frac{1}{x(x+1)} - \frac{x-1}{2!x^2(2x+1)} - \frac{(x-1)(2x-1)}{3!(x^3(3x+1))} - \frac{(x-1)(2x-1)(3x-1)}{4!x^4(4x+1)} - \dots$
is convergent for every positive x. Denoting its sum by $F(x)$, find $\lim_{x \to +0} F(x)$ and $\lim_{x \to \infty} F(x)$.

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5 In a lake there are several sorts of fish, in the following distribution: 18% catfish, 2% sturgeon and 80% other. Of a catch of ten fishes, let x denote the number of the catfish and y that of the sturgeons. Find the expectation of $\frac{x}{y+1}$

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6 In lawn-tennis the player who scores at least four points, while his opponent scores at least two points less, wins a game. The player who wins at least six games, while his opponent wins at least two games less, wins a set. What minimum percentage of all points does the winner have to score in a set?

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7 Let $f(x)$ be a polynomial with the following properties:
(i) $f(0)=0$; (ii) $\frac{f(a)-f(b)}{a-b}$ is an integer for any two different integers a and b. Is there a polynomial which has these properties, although not all of its coefficients are integers?

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8 Given a positive integer $n>3$, prove that the least common multiple of the products $x_1x_2\cdots x_k$ ($k\geq 1$) whose factors x_i are positive integers with $x_1+x_2+\cdots+x_k\leq n$, is less than $n!$.

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9 Let $\{m_1,m_2,\dots\}$ be a (finite or infinite) set of positive integers. Consider the system of congruences

(1) $x\equiv 2m_i^2 \pmod{2m_i-1}$ ($i=1,2,...$).

Give a necessary and sufficient condition for the system (1) to be solvable.

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10 Let $f(x)$ be a polynomial with integer coefficients and let p be a prime. Denote by z_1, ..., z_{p-1} the $(p-1)$th complex roots of unity. Prove that $f(z_1)\cdots f(z_{p-1})\equiv f(1)\cdots f(p-1) \pmod{p}$.

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11 Prove that, for every pair n, r of positive integers, there can be found a polynomial $f(x)$ of degree n with integer coefficients, so that every polynomial $g(x)$ of degree at most n, for which the coefficients of the polynomial $f(x)-g(x)$ are integers with absolute value not greater than r, is irreducible over the field of rational numbers.

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12 By number-theoretical functions, we will understand integer-valued functions defined on the set of all integers. Are there number-theoretical functions $f_0(x),f_1(x),f_2(x),\dots$ such that every number theoretical function $F(x)$ can be uniquely represented in the form $F(x)=\sum_{k=0}^{\infty} a_kf_k(x)$, $a_0,a_1,a_2,\dots$ being integers?

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13 Of how many terms does the expansion of a determinant of order $2n$ consist if those and only those elements a_{ik} are non-zero for which $i-k$ is divisible by n?

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14 For which commutative finite groups is the product of all elements equal to the unit element?

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15 Let the line $z=x,\ y=0$
rotate at a constant speed about the z-axis; let at the same time the point of intersection of this line with the z-axis be displaced along the z-axis at constant speed.
(a) Determine that surface of rotation upon which the resulting helical surface can be developed (i.e. isometrically mapped).
(b) Find those lines of the surface of rotation into which the axis and the generators of the helical surface will be mapped by this development.

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16 Let $\mathcal{F}$ be a surface which is simply covered by two systems of geodesics such that any two lines belonging to different systems form angles of the same opening. Prove that $\mathcal{F}$ can be developed (that is, isometrically mapped) into the plane.

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17 Let α be a projective plane and c a closed polygon on α. Prove that α will be decomposed into two regions by c if and only if there exists a straight line g in α which has an even number of points in common with c.
